Chap. 3 Rigid Bodies: Equivalent Systems of Forces

- Treatment of a body as a single particle is not always possible. In general, the size of the body and the specific points of application of the forces must be considered.

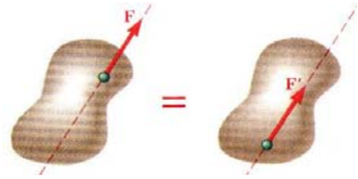
- To fully describe the effect of forces exerted on a rigid body, also need to consider:
  - moment of a force about a point
  - moment of a force about an axis
  - moment due to a couple

- Any system of forces acting on a rigid body can be replaced by an equivalent system consisting of one force acting at a given point and one couple.

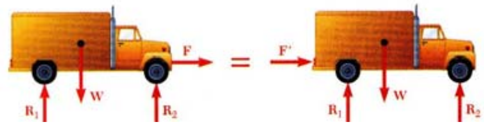
External/Internal Forces; Equivalent Forces

- **External forces** are shown in a free body diagram. **Internal forces** should not appear on a free body diagram.

- *Principle of Transmissibility* - Conditions of equilibrium or motion are not affected by *transmitting* a force along its line of action.
NOTE: **F** and **F'** are equivalent forces.

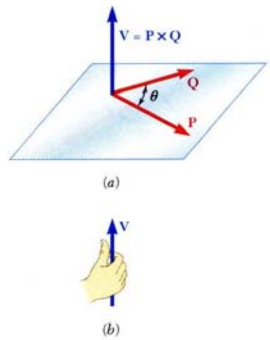


- Moving the point of application of the force **F** to the rear bumper does not affect the motion or the other forces acting on the truck.



Vector Product of Two Vectors

- Vector product of two vectors ***P*** and ***Q*** (a concept needed for moment) is defined as the vector ***V*** which satisfies the following conditions:
  1. Line of action of ***V*** is perpendicular to plane containing ***P*** and ***Q***.
  2. Magnitude of ***V*** is $V = PQ \sin\theta$
  3. Direction of ***V*** is obtained from the right-hand rule.

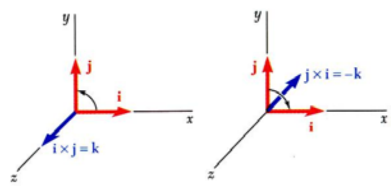


- Vector products:
  - are not commutative, $\boldsymbol{Q}\times\boldsymbol{P} = -(\boldsymbol{P}\times\boldsymbol{Q})$
  - are distributive, $\boldsymbol{P}\times(\boldsymbol{Q}_1+\boldsymbol{Q}_2) = \boldsymbol{P}\times\boldsymbol{Q}_1 + \boldsymbol{P}\times\boldsymbol{Q}_2$
  - are not associative, $(\boldsymbol{P}\times\boldsymbol{Q})\times\boldsymbol{S} \neq \boldsymbol{P}\times(\boldsymbol{Q}\times\boldsymbol{S})$

Vector Products: Rectangular Components

- Vector products of Cartesian unit vectors,

$$\begin{array}{lll} \vec{i}\times\vec{i}=0 & \vec{j}\times\vec{i}=-\vec{k} & \vec{k}\times\vec{i}=\vec{j} \\ \vec{i}\times\vec{j}=\vec{k} & \vec{j}\times\vec{j}=0 & \vec{k}\times\vec{j}=-\vec{i} \\ \vec{i}\times\vec{k}=-\vec{j} & \vec{j}\times\vec{k}=\vec{i} & \vec{k}\times\vec{k}=0 \end{array}$$

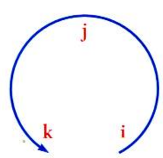


- Vector products in terms of rectangular coordinates

$$\vec{V} = \left(P_x\vec{i}+P_y\vec{j}+P_z\vec{k}\right)\times\left(Q_x\vec{i}+Q_y\vec{j}+Q_z\vec{k}\right)$$

$$= \left(P_yQ_z - P_zQ_y\right)\vec{i} + \left(P_zQ_x - P_xQ_z\right)\vec{j} + \left(P_xQ_y - P_yQ_x\right)\vec{k}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix}$$

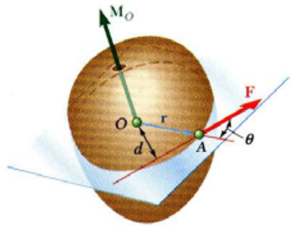


Moment of a Force About a Point

- A force vector is defined by its magnitude and direction. Its effect on the rigid body also depends on its point of application.
- The *moment* of ***F*** about *O* is defined as

$$\boldsymbol{M}_O = \boldsymbol{r} \times \boldsymbol{F}$$

- The moment vector $\boldsymbol{M}_O$ is perpendicular to the plane containing *O* and the force ***F***.
- Magnitude of $\boldsymbol{M}_O$, $M_O = rF \sin\theta = Fd$, measures the tendency of the force to cause rotation of the body about an axis along $\boldsymbol{M}_O$. The sense of the moment may be determined by the right-hand rule.
- Any force ***F'*** that has the same magnitude and direction as ***F***, is *equivalent* if it also has the same line of action and therefore, produces the same moment.

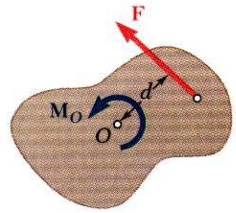


(*a*)

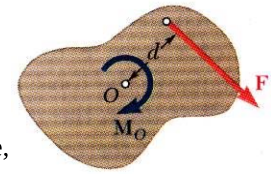
(*b*)

Moment of a Force About a Point

- *Two-dimensional structures* have length and breadth but negligible depth and are subjected to forces contained only in the plane of the structure.
- The plane of the structure contains the point *O* and the force ***F***. $\boldsymbol{M}_O$, the moment of the force about *O* is perpendicular to the plane.
- If the force tends to rotate the structure clockwise, the sense of the moment vector is out of the plane of the structure and the magnitude of the moment is positive.
- If the force tends to rotate the structure counterclockwise, the sense of the moment vector is into the plane of the structure and the magnitude of the moment is negative.



(*a*) $\boldsymbol{M}_O = + Fd$



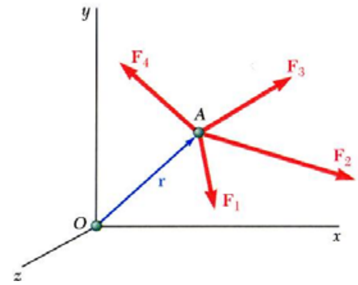
(*b*) $\boldsymbol{M}_O = - Fd$

Varignon's Theorem

- The moment about a give point *O* of the resultant of several concurrent forces is equal to the sum of the moments of the various moments about the same point *O*.

$$\vec{r}\times\left(\vec{F}_1+\vec{F}_2+\cdots\right)=\vec{r}\times\vec{F}_1+\vec{r}\times\vec{F}_2+\cdots$$

- Varignon's Theorem makes it possible to replace the direct determination of the moment of a force ***F*** by the moments of two or more component forces of ***F***.

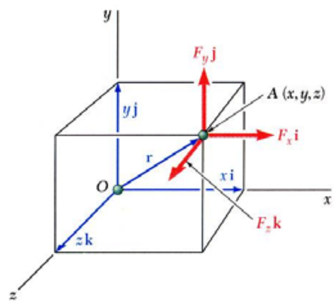


Rectangular Components of the Moment of a Force

The moment of ***F*** about *O*,

$$\vec{M}_O=\vec{r}\times\vec{F},\qquad \vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$$

$$\vec{F}=F_x\vec{i}+F_y\vec{j}+F_z\vec{k}$$



$$\vec{M}_O=M_x\vec{i}+M_y\vec{j}+M_z\vec{k}\quad=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\x&y&z\\F_x&F_y&F_z\end{vmatrix}$$

$$=\left(yF_z-zF_y\right)\vec{i}+\left(zF_x-xF_z\right)\vec{j}+\left(xF_y-yF_x\right)\vec{k}$$

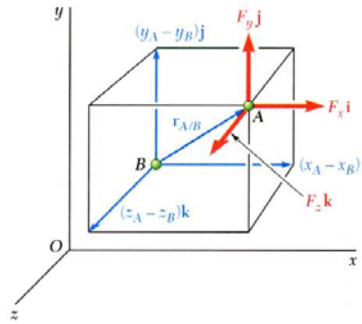
The components of $\vec{M}_o$, M_x, M_y, and M_z, represent the moments about the x-, y- and z-axis, respectively.

Rectangular Components of the Moment of a Force

The moment of ***F*** about *B*,

$$\vec{M}_B = \vec{r}_{A/B} \times \vec{F}$$

$$\vec{r}_{A/B} = \vec{r}_A - \vec{r}_B$$
$$= (x_A - x_B)\vec{i} + (y_A - y_B)\vec{j} + (z_A - z_B)\vec{k}$$
$$\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$$



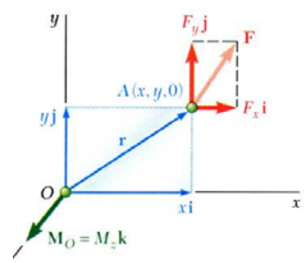
$$\vec{M}_B = \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ (x_A - x_B) & (y_A - y_B) & (z_A - z_B) \\ F_x & F_y & F_z \end{pmatrix}$$

Rectangular Components of the Moment of a Force

For two-dimensional structures,

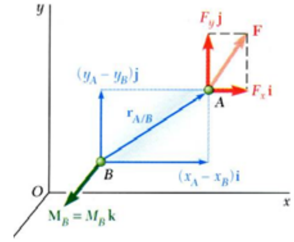
$$\vec{M}_O = (xF_y - yF_z)\vec{k}$$

$$M_O = M_Z$$
$$= xF_y - yF_z$$

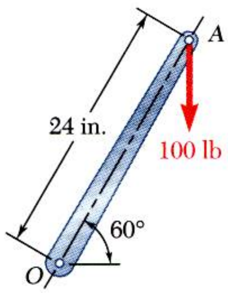


$$\vec{M}_B = \left[(x_A - x_B)F_y - (y_A - y_B)F_z\right]\vec{k}$$

$$M_B = M_Z$$
$$= (x_A - x_B)F_y - (y_A - y_B)F_z$$



Sample Problem 3.1



A 100-lb vertical force is applied to the end of a lever which is attached to a shaft (not shown) at *O*.

Determine:

a) the moment about *O*,

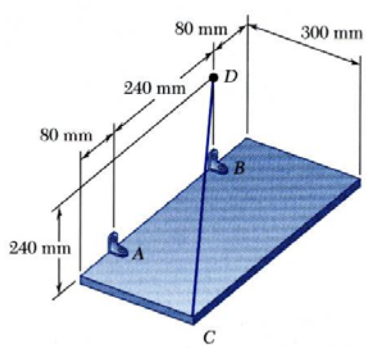
b) the horizontal force at *A* which creates the same moment,

c) the smallest force at A which produces the same moment,

d) the location for a 240-lb vertical force to produce the same moment,

e) whether any of the forces from b, c, and d is equivalent to the original force.

Sample Problem 3.4



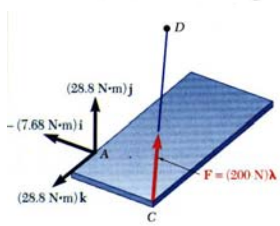
The rectangular plate is supported by the brackets at *A* and *B* and by a wire *CD*. Knowing that the tension in the wire is 200 N, determine the moment about *A* of the force exerted by the wire at *C*.

$$\vec{M}_A = \vec{r}_{C/A} \times \vec{F}$$

$$\vec{r}_{C/A} = \vec{r}_C - \vec{r}_A = (0.3\text{ m})\vec{i} + (0.08\text{ m})\vec{j}$$

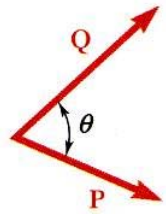
$$\vec{F} = (200\text{ N})\frac{-(0.3\text{ m})\vec{i} + (0.24\text{ m})\vec{j} - (0.32\text{ m})\vec{k}}{\sqrt{0.3^2 + 0.24^2 + 0.32^2}\text{ m}}$$

$$= -(120\text{ N})\vec{i} + (96\text{ N})\vec{j} - (128\text{ N})\vec{k}$$



$$\vec{M}_A = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.3 & 0 & 0.08 \\ -120 & 96 & -128 \end{vmatrix} = -(7.68\text{ N}\cdot\text{m})\vec{i} + (28.8\text{ N}\cdot\text{m})\vec{j} + (28.8\text{ N}\cdot\text{m})\vec{k}$$

Scalar Product of Two Vectors



- The *scalar product* or *dot product* between two vectors ***P*** and ***Q*** is defined as

  $\vec{P}\bullet\vec{Q} = PQ\cos\theta \quad (\text{scalar result})$

- Scalar products:
  - are commutative, $\vec{P}\bullet\vec{Q} = \vec{Q}\bullet\vec{P}$
  - are distributive, $\vec{P}\bullet\left(\vec{Q}_1+\vec{Q}_2\right) = \vec{P}\bullet\vec{Q}_1 + \vec{P}\bullet\vec{Q}_2$
  - are not associative, $\left(\vec{P}\bullet\vec{Q}\right)\bullet\vec{S} = \text{undefined}$

- Scalar products with Cartesian unit components,

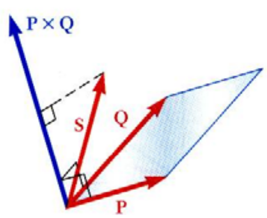
$$\vec{P}\bullet\vec{Q} = \left(P_x\vec{i} + P_y\vec{j} + P_z\vec{k}\right)\bullet\left(Q_x\vec{i} + Q_y\vec{j} + Q_z\vec{k}\right)$$

$$\vec{i}\bullet\vec{i} = 1 \quad \vec{j}\bullet\vec{j} = 1 \quad \vec{k}\bullet\vec{k} = 1 \quad \vec{i}\bullet\vec{j} = 0 \quad \vec{j}\bullet\vec{k} = 0 \quad \vec{k}\bullet\vec{i} = 0$$

$$\vec{P}\bullet\vec{Q} = P_xQ_x + P_yQ_y + P_zQ_z$$

$$\vec{P}\bullet\vec{P} = P_x^2 + P_y^2 + P_z^2 = P^2$$

Mixed Triple Product of Three Vectors



- Mixed triple product of three vectors,

  $\vec{S}\bullet\left(\vec{P}\times\vec{Q}\right) = \text{scalar result}$

- The six mixed triple products formed from ***S***, ***P***, and ***Q*** have equal magnitudes but not the same sign,

$$\vec{S}\bullet\left(\vec{P}\times\vec{Q}\right) = \vec{P}\bullet\left(\vec{Q}\times\vec{S}\right) = \vec{Q}\bullet\left(\vec{S}\times\vec{P}\right)$$
$$= -\vec{S}\bullet\left(\vec{Q}\times P\right) = -\vec{P}\bullet\left(\vec{S}\times\vec{Q}\right) = -\vec{Q}\bullet\left(\vec{P}\times\vec{S}\right)$$

$$\vec{S}\bullet\left(\vec{P}\times\vec{Q}\right) = S_x\left(P_yQ_z - P_zQ_y\right) + S_y\left(P_zQ_x - P_xQ_z\right)$$
$$+ S_z\left(P_xQ_y - P_yQ_x\right)$$
$$= \begin{pmatrix} S_x & S_y & S_z \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{pmatrix}$$

Moment of a Force About a Given Axis

- Moment $\boldsymbol{M_O}$ of a force $\boldsymbol{F}$ applied at the point $\boldsymbol{A}$ about a point $\boldsymbol{O}$,

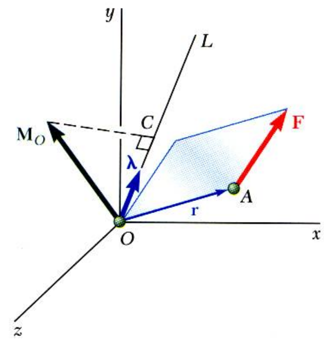
$$\vec{M}_O = \vec{r} \times \vec{F}$$

- Scalar moment M_{OL} about an axis $\boldsymbol{OL}$ is the projection of the moment vector $\boldsymbol{M_O}$ onto the axis,

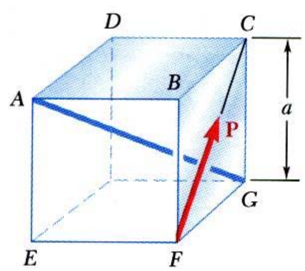
$$M_{OL} = \vec{\lambda} \bullet \vec{M}_O = \vec{\lambda} \bullet \left(\vec{r} \times \vec{F}\right)$$

- Moments of $\boldsymbol{F}$ about the coordinate axes,

$$M_x = yF_z - zF_y$$
$$M_y = zF_x - xF_z$$
$$M_z = xF_y - yF_x$$



Sample Problem 3.5



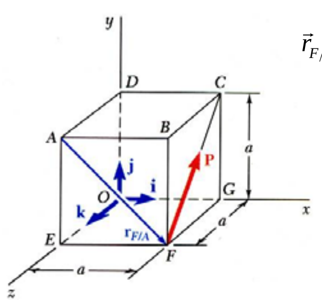
A cube is acted on by a force $\boldsymbol{P}$ as shown. Determine the moment of $\boldsymbol{P}$

a) about A $\qquad \vec{M}_A = \vec{r}_{F/A} \times \vec{P}$

b) about the edge AB and

c) about the diagonal AG of the cube.

d) Determine the perpendicular distance between AG and FC.



$$\vec{r}_{F/A} = a\vec{i} - a\vec{j} = a\left(\vec{i} - \vec{j}\right) \qquad \vec{P} = P\left(\sqrt{2}\,\vec{i} + \sqrt{2}\vec{j}\right) = P\sqrt{2}\left(\vec{i} + \vec{j}\right)$$

$$\boxed{\vec{M}_A = \left(aP\sqrt{2}\right)\left(\vec{i} + \vec{j} + \vec{k}\right)}$$

$$M_{AB} = \vec{i} \bullet \vec{M}_A = aP\sqrt{2}$$

$$M_{AG} = \vec{\lambda}_{AG} \bullet \vec{M}_A \qquad \vec{\lambda}_{AG} = \frac{1}{\sqrt{3}}\left(\vec{i} - \vec{j} - \vec{k}\right)$$

$$M_{AG} = -\frac{aP}{\sqrt{6}}$$

$$\boxed{d = \frac{a}{\sqrt{6}}}$$

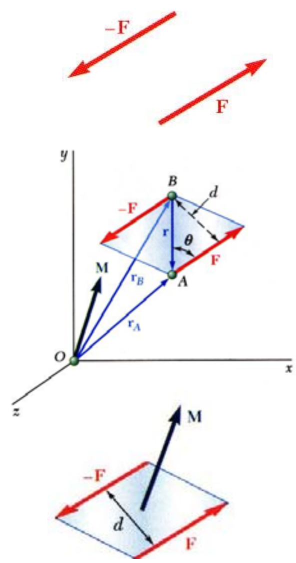
Moment of a Couple

- Two forces **F** and **-F** having the same magnitude, parallel lines of action, and opposite sense are said to form a *couple*.

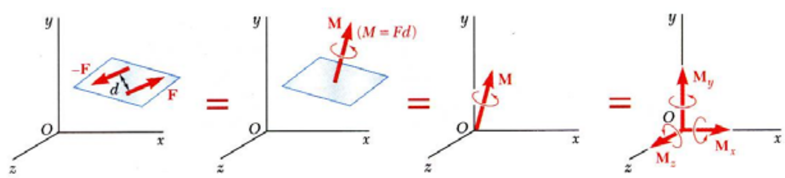
- Moment of the couple,

$$\vec{M} = \vec{r}_A \times \vec{F} + \vec{r}_B \times \left(-\vec{F}\right)$$
$$= \left(\vec{r}_A - \vec{r}_B\right) \times \vec{F}$$
$$= \vec{r} \times \vec{F}$$
$$M = rF \sin\theta = Fd$$

- The moment vector of the couple is independent of the choice of the origin of the coordinate axes, i.e., it is a *free vector* that can be applied at any point with the same effect.



Couples Can Be Represented by Vectors



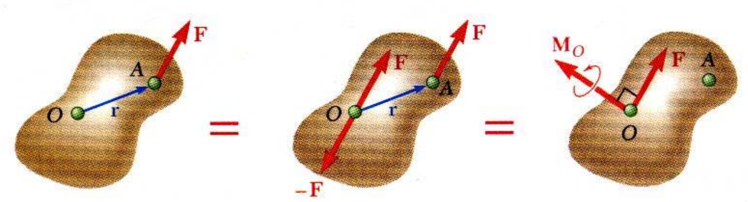
- A couple can be represented by a vector with magnitude and direction equal to the moment of the couple.

- *Couple vectors* obey the law of addition of vectors.

- Couple vectors are free vectors, i.e., there is no point of application – it simply acts on the body.

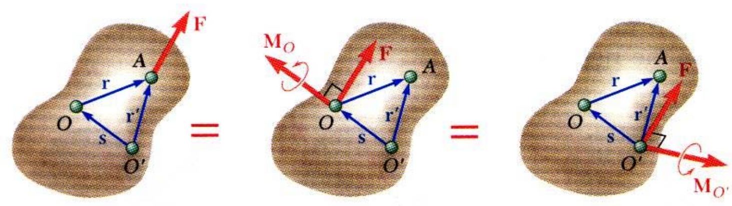
- Couple vectors may be resolved into component vectors.

Resolution of a Force Into a Force at *O* and a Couple



- Force vector ***F*** can not be simply moved to *O* without modifying its action on the body. Attaching equal and opposite force vectors at *O* produces no net effect on the body. The three forces may be replaced by an equivalent force vector and couple vector, i.e, a *force-couple system*.

Resolution of a Force Into a Force at *O* and a Couple



- Moving ***F*** from *A* to a different point *O'* requires the addition of a different couple vector ***M**$_{O'}$*

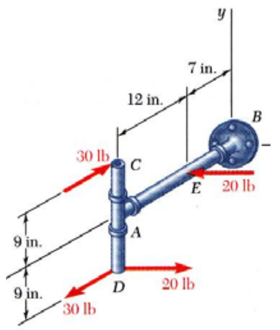
$$\vec{M}_{O'} = \vec{r}' \times \vec{F}$$

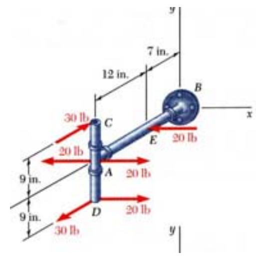
- The moments of ***F*** about O and *O'* are related,

$$\vec{M}_{O'} = \vec{r}' \times \vec{F} = (\vec{r} + \vec{s}) \times \vec{F} = \vec{r} \times \vec{F} + \vec{s} \times \vec{F}$$
$$= \vec{M}_O + \vec{s} \times \vec{F}$$

- Moving the force-couple system from *O* to *O'* requires the addition of the moment of the force at *O* about *O'*.

Sample Problem 3.6





Attach equal and opposite 20 lb forces in the $\pm x$ direction at A

Determine the components of the single couple equivalent to the couples shown.

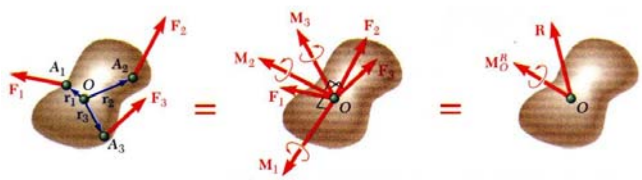
$$M_x = -(30 \text{ lb})(18 \text{ in.}) = -540 \text{ lb} \cdot \text{in.}$$

$$M_y = +(20 \text{ lb})(12 \text{ in.}) = +240 \text{lb} \cdot \text{in.}$$

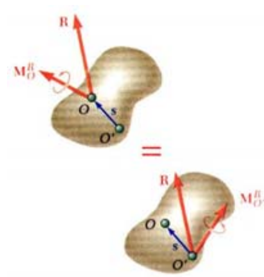
$$M_z = +(20 \text{ lb})(9 \text{ in.}) = +180 \text{ lb} \cdot \text{in.}$$

$$\vec{M} = -(540 \text{ lb} \cdot \text{in.})\vec{i} + (240 \text{lb} \cdot \text{in.})\vec{j} + (180 \text{ lb} \cdot \text{in.})\vec{k}$$

System of Forces: Reduction to a Force and Couple



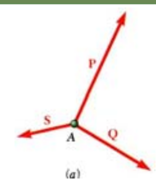
- A system of forces may be replaced by a collection of force-couple systems acting at a given point O
- The force and couple vectors may be combined into a resultant force vector and a resultant couple vector,
  $$\vec{R} = \sum \vec{F} \vec{M}_O^R = \sum (\vec{r} \times \vec{F})$$
- The force-couple system at O may be moved to O' with the addition of the moment of $\boldsymbol{R}$ about O' ,
  $$\vec{M}_{O\text{n}}^R = \vec{M}_O^R + \vec{s} \times \vec{R}$$
- Two systems of forces are equivalent if they can be reduced to the same force-couple system.

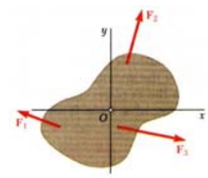


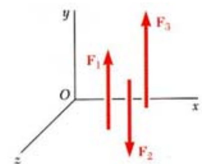
Further Reduction of a System of Forces

- If the resultant force and couple at O are mutually perpendicular, they can be replaced by a single force acting along a new line of action.

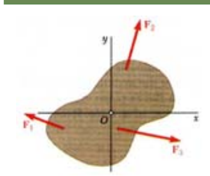
- The resultant force-couple system for a system of forces will be mutually perpendicular if:
  1) the forces are concurrent,
  2) the forces are coplanar, or
  3) the forces are parallel.

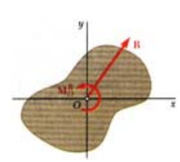


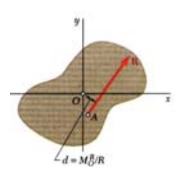




Further Reduction of a System of Forces





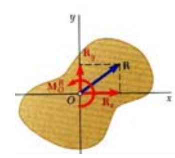


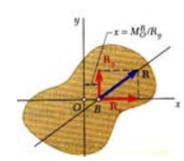
- System of coplanar forces is reduced to a force-couple system $\vec{R}$ and $\vec{M}_O^R$ that is mutually perpendicular.

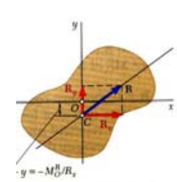
- System can be reduced to a single force by moving the line of action of $\vec{R}$ until its moment about O becomes $\vec{M}_O^R$

- In terms of rectangular coordinates,

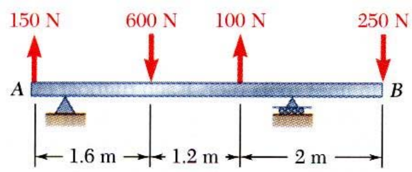
$$xR_y - yR_x = M_O^R$$

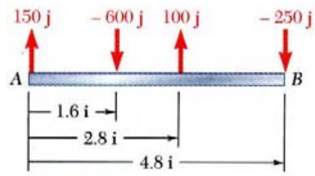






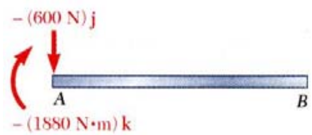
Sample Problem 3.8





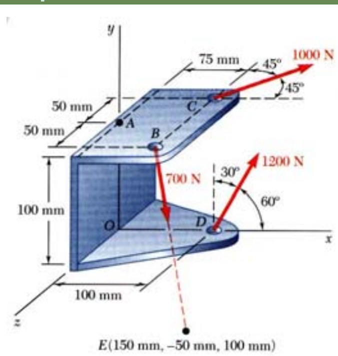
For the beam, reduce the system of forces shown to (a) an equivalent force-couple system at *A*, (b) an equivalent force couple system at *B*, and (c) a single force or resultant.

Note: Since the support reactions are not included, the given system will not maintain the beam in equilibrium.



Sanity check

Sample Problem 3.10



Three cables are attached to the bracket as shown. Replace the forces with an equivalent force-couple system at *A*.

Solution by brute force:

$$\vec{r}_{B/A} = 0.075\vec{i} + 0.050\vec{k}\ (\mathrm{m})$$

$$\vec{r}_{C/A} = 0.075\vec{i} - 0.050\vec{k}\ (\mathrm{m})$$

$$\vec{r}_{D/A} = 0.100\vec{i} - 0.100\vec{j}\ (\mathrm{m})$$

$$\vec{F}_B = 300\vec{i} - 600\vec{j} + 200\vec{k}\quad (\mathrm{N})$$

$$\vec{F}_C = 707\vec{i} - 707\vec{j}\quad (\mathrm{N})$$

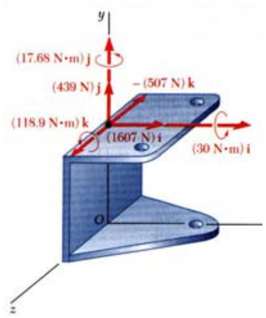
$$\vec{F}_D = 600\vec{i} + 1039\vec{j}\quad (\mathrm{N})$$

Sample Problem 3.10

- Compute the equivalent force,

$$\vec{R} = \sum \vec{F}$$
$$= (300 + 707 + 600)\vec{i} + (-600 + 1039)\vec{j} + (200 - 707)\vec{k}$$

$$\boxed{\vec{R} = 1607\vec{i} + 439\vec{j} - 507\vec{k}\ \ (\mathrm{N})}$$



- Compute the equivalent couple,

$$\vec{M}_A^R = \sum (\vec{r} \times \vec{F})$$

$$\vec{r}_{B/A} \times \vec{F}_B = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.075 & 0 & 0.050 \\ 300 & -600 & 200 \end{vmatrix} = 30\vec{i} - 45\vec{k}$$

$$\vec{r}_{C/A} \times \vec{F}_c = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.075 & 0 & -0.050 \\ 707 & 0 & -707 \end{vmatrix} = 17.68\vec{j}$$

$$\vec{r}_{D/A} \times \vec{F}_D = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.100 & -0.100 & 0 \\ 600 & 1039 & 0 \end{vmatrix} = 163.9\vec{k}$$

$$\boxed{\vec{M}_A^R = 30\vec{i} + 17.68\vec{j} + 118.9\vec{k}}$$