Chapter 10: Rotational Motion and Angular Momentum

Average angular velocity = (angular displacement) / (elaspsed time)	$\overline{\omega} = \frac{\Delta\theta}{\Delta t}$
Instantaneous angular velocity	$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}$
Average angular acceleration = (change in angular velocity) / (elaspsed time)	$\overline{\alpha} = \frac{\Delta\omega}{\Delta t}$
Instantaneous angular acceleration	$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t}$

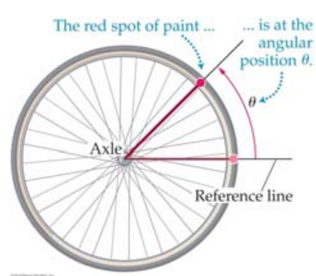
Unit of angular velocity: rad/sec
Unit of angular acceleration: rad/sec^2

Rotational Kinetic Energy

We know that the kinetic energy of a point object with a mass of m and a velocity/speed of v is

$$K = \tfrac{1}{2} m v^2$$

How about the kinetic energy of an extended object with a total mass of m rotating about a fixed axis with a constant angular velocity of ω?



There should not be two different definitions of the same thing! An extended object is nothing but a collection of small "point-like" objects, each with a tiny bit of mass m_i and each with a constant speed of v_i. The total kinetic energy for the rotating body should be just all of these tiny little $K_i = \tfrac{1}{2} m_i v_i^2$'s put together.

Rotational Kinetic Energy and Moment of Inertia

For a rigid system consisting of many particles,

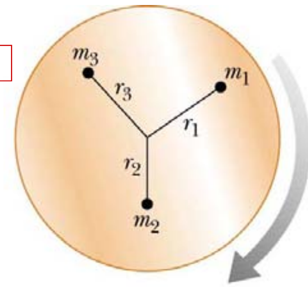
$$K_1 = \frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 (r_1^2 \omega^2)$$

r: lever arm

$$K_2 = \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_2 r_2^2 \omega^2$$

$$\ldots \qquad \ldots$$

$$K_{tot} = \sum_i \frac{1}{2} m_i v_i^2 = \frac{\omega^2}{2} \sum_i m_i r_i^2$$



The moment of inertia for a (rigid) system consisting of many particles is found to be

Note: distance to the axis of rotation (a line)! Distance parallel to the axis does not matter.

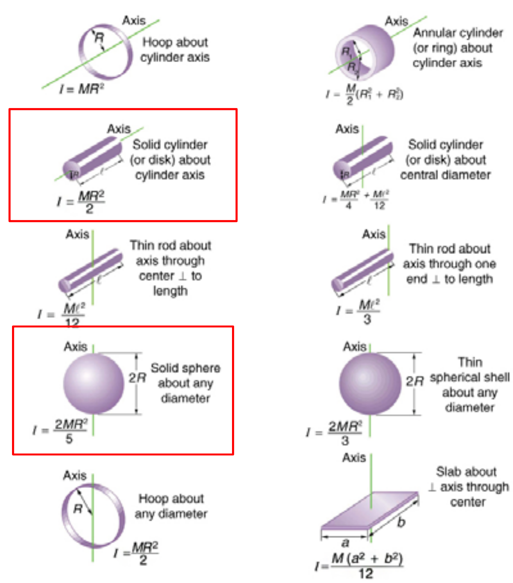
$$K_{ROT} = \frac{1}{2} I \omega^2 \qquad I = \sum mr^2$$

Obviously, the moment of inertia of an object depends on the chosen axis of rotation.

Moments of Inertia

$$I = \sum mr^2$$

Moment of inertia of an object depends on the location of the rotational axis!



Rotational Work And Energy

The **rotational work** W_R done by a constant torque τ in turning an object through an angle $\Delta\theta$ (expressed in radians) is

$$W_R = \tau\, \Delta\theta$$

Note that, just like linear work before, the rotational work could be negative.

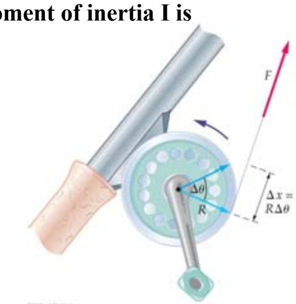
The **rotational kinetic energy** K_R of a rigid object rotating with an angular speed ω about a fixed axis and having a moment of inertia I is

$$K_R = I\, \omega^2 /2$$

Rotational Work-Energy Theorem:

$$W_R = \Delta K_R$$

Unit for both W_R and KE_R is joule (J).



Total Mechanical Energy

The total kinetic energy of an object that moves with both a linear velocity and a rotational velocity is the sum of the two contributions. Here *v* is the linear velocity of the center of mass of the object and ω is the angular velocity about its center of mass.

$$\boxed{K_{Total} = \frac{1}{2}mv^2 + \frac{1}{2}I\,\omega^2}$$

of center of mass

The total mechanical energy of the object is then the sum of its kinetic energy and potential energy. In the absence of work due to non-conservative forces, the total mechanical energy is conserved.

of center of mass

$$\boxed{E_{Total} = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}kx^2 + mgh}$$

Angular Acceleration Due To Torque

If a torque τ is applied for an angular displacement Δθ and it leads the angular velocity to change from ω_i to ω_f, what is the average angular acceleration α for this torque during this period of time?

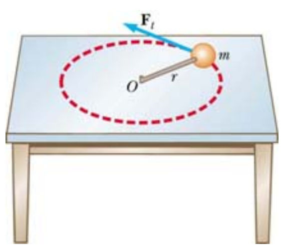
$$W = \tau \Delta\theta = \tfrac{1}{2} I \omega_f^2 - \tfrac{1}{2} I \omega_i^2$$

$$\omega_f^2 - \omega_i^2 = 2\Delta\theta \frac{\tau}{I}$$

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t}$$

$$\boxed{= \alpha \quad ?}$$

Consequence of Net Torques: Angular Accel.



For a rigid body rotating about a fixed axis

Net external torque = (moment of intertia) * (angular acceleration)

$$\Sigma \tau = I \alpha$$

Example: For a point object rotating about an axis a distance r away from the object,

$$F_T = ma = mr\alpha \qquad rF_T = \tau = mr^2\alpha = I\alpha$$

because the moment of inertia of a point object going around an external is

$$I = mr^2$$

“Rotational Collision”: What is Conserved?

In a physics 1100 lab experiment, a disk with a moment of inertia of I_1 and an initial angular velocity ω_1 is dropped concentrically onto a disk with I_2 and ω_2. In the absence of resistance and friction from external sources, what angular quantity, if any, is guaranteed to be conserved?

Because of Newton’s third law, whatever force #1 gives #2 at time t and position r must be countered with a negative force from #2 to #1, at the same position and time. Therefore, the torque #2 gives #1, τ_{12} is negative of the torque #1 gives #2, τ_{21}, at all times: $\tau_{12} = - \tau_{21}$. Since the only torque each disk receives is from each other, one gets

$$\Delta\omega_1 = \alpha_1 \Delta t = (\tau_{12} / I_1)\Delta t \qquad \Delta\omega_2 = \alpha_2 \Delta t = (\tau_{21} / I_2)\Delta t$$

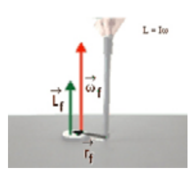
Combining, one gets $I_1 \Delta\omega_1 + I_2 \Delta\omega_2 = (\tau_{12} + \tau_{21})\Delta t = 0$

$$\Delta(I_1\omega_1 + I_2\omega_2) = 0 \quad \Rightarrow \quad I_1\omega_1 + I_2\omega_2 = const.$$

Angular Momentum

The angular momentum L of a rigid body rotating about a fixed axis is the product of the body’s moment of inertia I and its angular velocity ω with respect to the axis:

$$\mathbf{L = I}\,\boldsymbol{\omega} \qquad \tau = \Delta L / \Delta t$$



Changes in angular momentum are due to

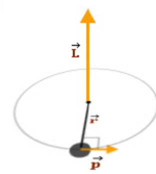
“rotational impulse” = $\tau\ \Delta t$

Internal forces of a system cannot lead to a net torque on the system, because of the action and the reaction act in opposite directions along the same line.

If the net external torque on a system is zero, the total angular momentum of the system remains constant, i.e. it is conserved.

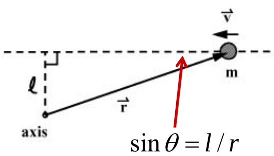
Angular Momentum of Moving Particle

What is the angular momentum of a particle with mass m moving with velocity v in a circular orbit of radius R about the origin? The moment of inertia of such a "body" is $I=mR^2$ and the angular velocity is $\omega=v/R$. The angular momentum is therefore



$$L = I\omega = (mR^2)(v/R) = R(mv) = Rp$$

What if the particle is moving along a straight line? There is still an angular momentum(!!) of L=mv$l$, where l is the "lever arm" of the straight line.



$$L = I\omega = (mr^2)(v\sin\theta/r) = mvl = lp$$

Note that the definitions of angular momentum for rigid bodies and for point objects are completely equivalent.

$$\vec{L} = \vec{r} \otimes \vec{p}$$

Rotational Analog of Translational Concepts

Item	Rotational	Translational
Displacement	θ	s
Velocity	ω	v
Acceleration	α	a
Cause of Accel.	Torque τ	Force F
Inertia	Moment of Inertia I	Mass m
Newton's 2nd Law	τ = Iα	F = ma
Work	τ θ	F s
Kinetic Energy	$I\omega^2 / 2$	$mv^2 / 2$
Momentum	Angular momentum Iω	Linear Momentum mv

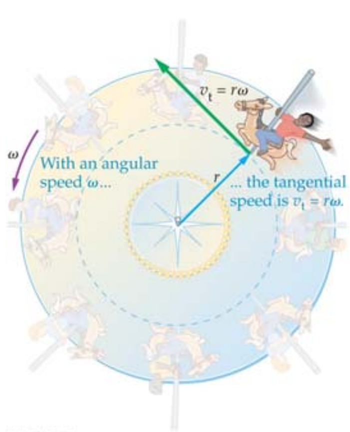
Equations of Rotational Kinematics

constant acceleration

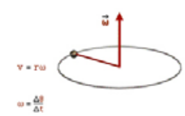
Rotational Motion	Quantity	Linear Motion
θ	Displacement	x
ω_o	Initial velocity	v_o
ω	Final velocity	v
α	Acceleration	a
t	Time	t

Rotational Motion (α = constant)	Linear Motion (a = constant)
$\omega = \omega_o + \alpha t$	$v = v_o + at$
$\Delta\theta = (\omega_o + \omega)\, t / 2$	$\Delta x = (v_o + v)\, t / 2$
$\Delta\theta = \omega_o t + \alpha t^2/2$	$\Delta x = v_o t + at^2 / 2$
$\omega^2 = \omega_o^2 + 2\, \alpha\, \Delta\theta$	$v^2 = v_o^2 + 2\, a\, \Delta x$

Connecting Rotational and Linear Quantities

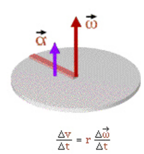


$\boxed{v_T = r\omega}$



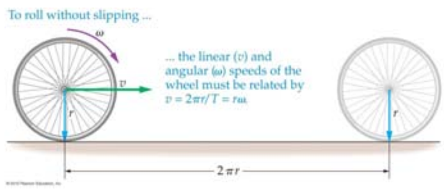
The tangential speed of a point on a rigid rotating object equals the distance of that point from the axis of rotation multiplied by the angular speed.

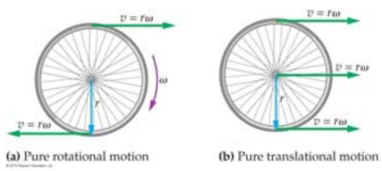
$\boxed{a_T = r\alpha}$



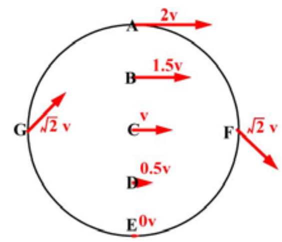
The tangential acceleration of a point on a rigid rotating object equals the distance of that point from the axis of rotation multiplied by the angular acceleration.

Rolling Motion





If the tire does not slip, v = rω.



Instantaneous speed at different points of a non-slip tire with a linear velocity of v. Note that the motion of the tire can be described as a perfect rotation about the point "E".

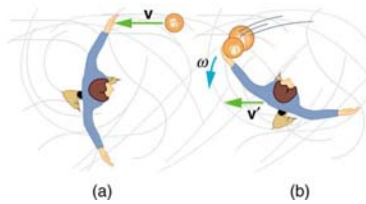
Examples

28. A ball with an initial velocity of 8.00 m/s rolls up a hill without slipping. Treating the all as a spherical shell, calculate the vertical height it reaches. (b) Repeat the calculation for the same ball if it slides up the hill without rolling.

40. Three children are riding on the edge of a merry-go-round that is 100 kg, has a 1.60-m radius, and is spinning at 20.0 rpm. The children have masses of 22.0, 28.0, and 33.0 kg. If the child who has a mass of 28.0 moves to the center of the merry-go-round, what is the new angular velocity in rpm?

Example

46. Suppose a 0.250-kg ball is thrown at 15.0 m/s to a motionless person standing on ice who catches it with an outstretched arm. (a) Calculate the final linear velocity of the person, given his mass is 70.0 kg. (b) What is his angular velocity if each arm is 5.00 kg? You may treat the ball as a point mass and treat the person's arms as uniform rods (each has a length of 0.900 m) and the rest of his gody as a uniform cylinder of radius 0.180 m. Neglect the effect of the all on his center of mass so that his center of mass remains in his geometrical center. (c) Compare the initial and final total kinetic energies.



Summary of Chapter 10

$$\theta \text{ (in radians)} = \text{arc length/radius} = s/r$$

- Average and instantaneous angular velocity and angular acceleration:

- Linear and angular equations of motion:

$$v_t = r\omega \qquad a_{cp} = r\omega^2 \qquad a_t = r\alpha$$

- Rolling motion: $\omega = v/r$
- Kinetic energy of rotation: $K = \frac{1}{2}I\omega^2$
- Moment of inertia: $I = \sum m_i r_i^2$
- Kinetic energy of an object rolling without slipping:

$$K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$