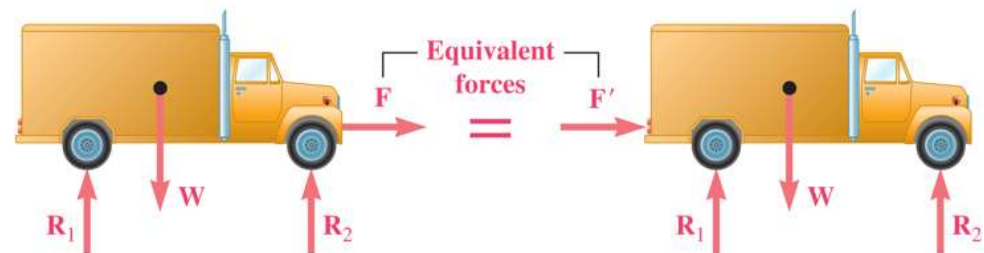
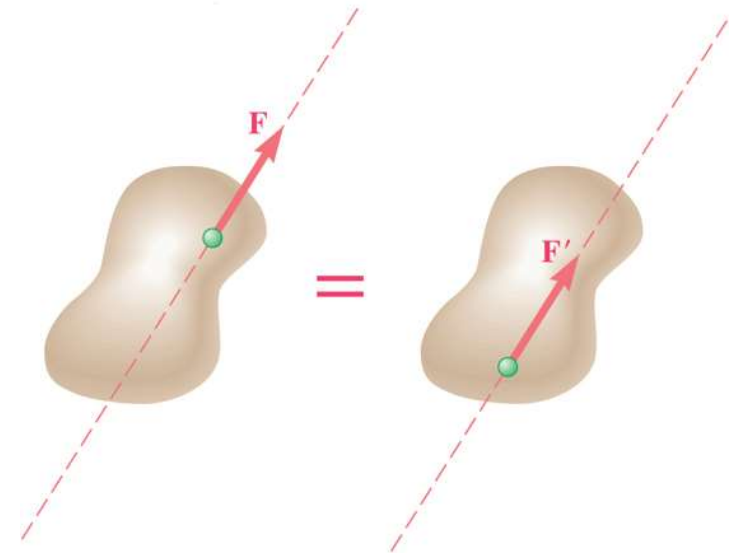


Chap. 3 Rigid Bodies: Equivalent Systems of Forces

- Treatment of a body as a single particle is not always possible. In general, the size of the body and the specific points of application of the forces must be considered.
- To fully describe the effect of forces exerted on a rigid body, also need to consider:
 - moment of a force about a point
 - moment of a force about an axis
 - moment due to a couple
- Any system of forces acting on a rigid body can be replaced by an equivalent system consisting of one force acting at a given point and one couple.

External/Internal Forces; Equivalent Forces

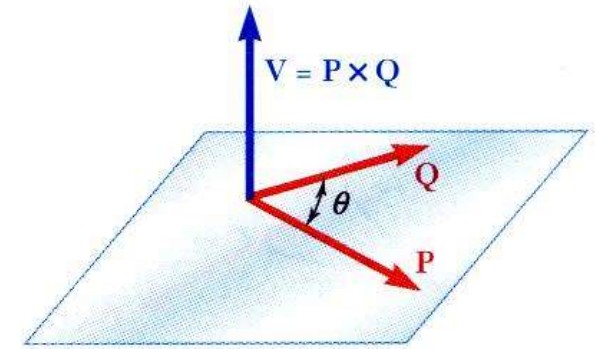
- **External forces** are shown in a free body diagram. **Internal forces** should not appear on a free body diagram.
- *Principle of Transmissibility* - Conditions of equilibrium or motion are not affected by *transmitting* a force along its line of action.
NOTE: F and F' are equivalent forces.
- Moving the point of application of the force F to the rear bumper does not affect the motion or the other forces acting on the truck.



Vector Product of Two Vectors

- Vector product of two vectors \mathbf{P} and \mathbf{Q} (a concept needed for moment) is defined as the vector \mathbf{V} which satisfies the following conditions:

1. Line of action of \mathbf{V} is perpendicular to plane containing \mathbf{P} and \mathbf{Q} .
2. Magnitude of \mathbf{V} is $V = PQ \sin \theta$
3. Direction of \mathbf{V} is obtained from the right-hand rule.



(a)



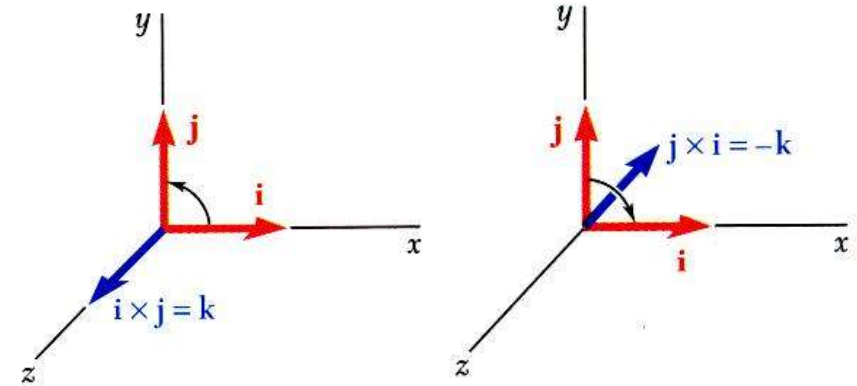
(b)

- Vector products:
 - are not commutative, $\mathbf{Q} \times \mathbf{P} = -(\mathbf{P} \times \mathbf{Q})$
 - are distributive, $\mathbf{P} \times (\mathbf{Q}_1 + \mathbf{Q}_2) = \mathbf{P} \times \mathbf{Q}_1 + \mathbf{P} \times \mathbf{Q}_2$
 - are not associative, $(\mathbf{P} \times \mathbf{Q}) \times \mathbf{S} \neq \mathbf{P} \times (\mathbf{Q} \times \mathbf{S})$

Vector Products: Rectangular Components

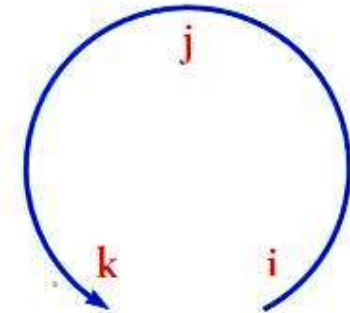
- Vector products of Cartesian unit vectors,

$$\begin{aligned} \vec{i} \times \vec{i} &= 0 & \vec{j} \times \vec{i} &= -\vec{k} & \vec{k} \times \vec{i} &= \vec{j} \\ \vec{i} \times \vec{j} &= \vec{k} & \vec{j} \times \vec{j} &= 0 & \vec{k} \times \vec{j} &= -\vec{i} \\ \vec{i} \times \vec{k} &= -\vec{j} & \vec{j} \times \vec{k} &= \vec{i} & \vec{k} \times \vec{k} &= 0 \end{aligned}$$



- Vector products in terms of rectangular coordinates

$$\begin{aligned} \vec{V} &= (P_x \vec{i} + P_y \vec{j} + P_z \vec{k}) \times (Q_x \vec{i} + Q_y \vec{j} + Q_z \vec{k}) \\ &= (P_y Q_z - P_z Q_y) \vec{i} + (P_z Q_x - P_x Q_z) \vec{j} \\ &\quad + (P_x Q_y - P_y Q_x) \vec{k} \end{aligned}$$



$$= \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{pmatrix}$$

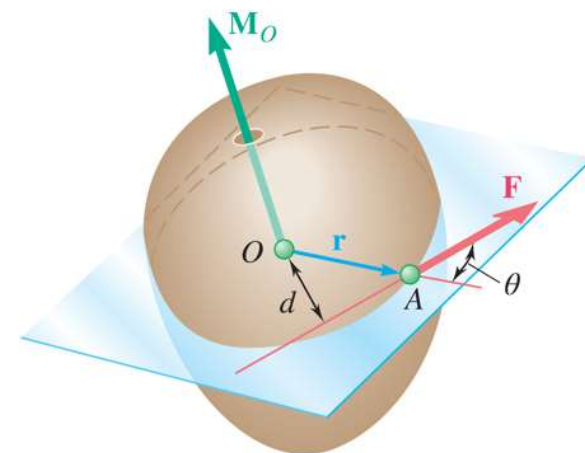
Moment of a Force About a Point

- A force vector is defined by its magnitude and direction. Its effect on the rigid body also depends on its point of application.

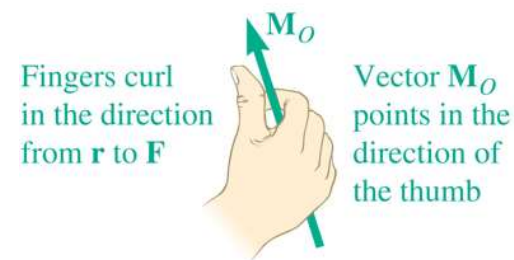
- The *moment* of F about O is defined as

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$$

- The moment vector \mathbf{M}_O is perpendicular to the plane containing O and the force F .
- Magnitude of \mathbf{M}_O , $M_O = rF \sin \theta = Fd$, measures the tendency of the force to cause rotation of the body about an axis along \mathbf{M}_O . The sense of the moment may be determined by the right-hand rule.
- Any force F' that has the same magnitude and direction as F , is *equivalent* if it also has the same line of action and therefore, produces the same moment.



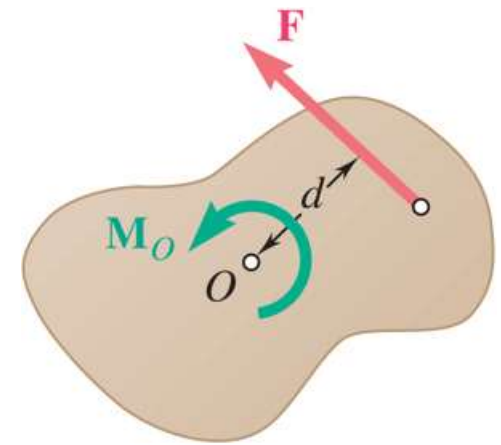
(a)



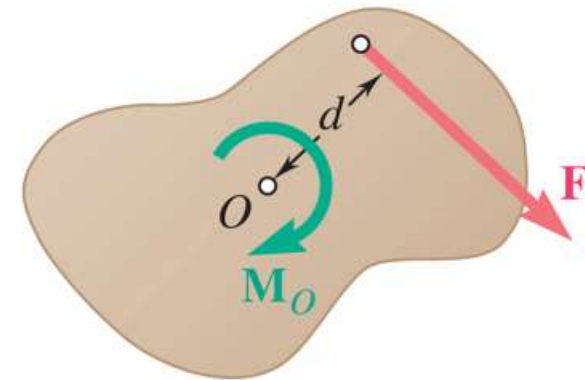
(b)

Moment of a Force About a Point

- *Two-dimensional structures* have length and breadth but negligible depth and are subjected to forces contained only in the plane of the structure.
- The plane of the structure contains the point O and the force F . M_O , the moment of the force about O is perpendicular to the plane.
- If the force tends to rotate the structure clockwise, the sense of the moment vector is out of the plane of the structure and the magnitude of the moment is positive.
- If the force tends to rotate the structure counterclockwise, the sense of the moment vector is into the plane of the structure and the magnitude of the moment is negative.



(a) $M_O = +Fd$



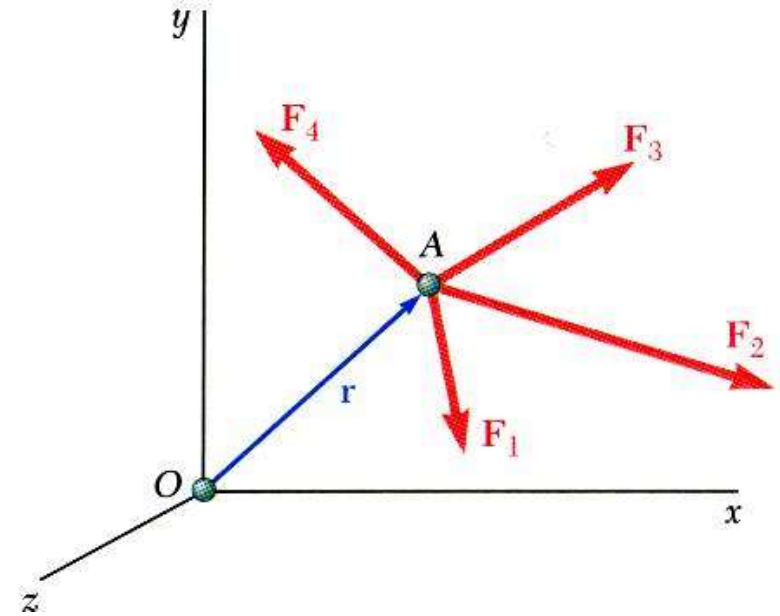
(b) $M_O = -Fd$

Varignon's Theorem

- The moment about a give point O of the resultant of several concurrent forces is equal to the sum of the moments of the various moments about the same point O .

$$\vec{r} \times (\vec{F}_1 + \vec{F}_2 + \dots) = \vec{r} \times \vec{F}_1 + \vec{r} \times \vec{F}_2 + \dots$$

- Varignon's Theorem makes it possible to replace the direct determination of the moment of a force \mathbf{F} by the moments of two or more component forces of \mathbf{F} .



Rectangular Components of the Moment of a Force

The moment of \vec{F} about O ,

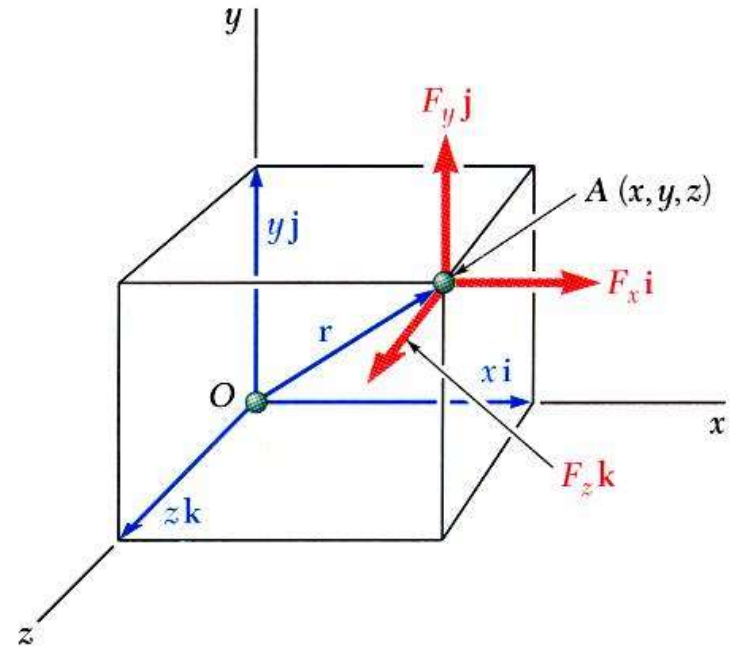
$$\vec{M}_O = \vec{r} \times \vec{F}, \quad \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$$

$$\vec{M}_O = M_x\vec{i} + M_y\vec{j} + M_z\vec{k} = \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ F_x & F_y & F_z \end{pmatrix}$$

$$= (yF_z - zF_y)\vec{i} + (zF_x - xF_z)\vec{j} + (xF_y - yF_x)\vec{k}$$

The components of \vec{M}_O , M_x , M_y , and M_z , represent the moments about the x-, y- and z-axis, respectively.



Rectangular Components of the Moment of a Force

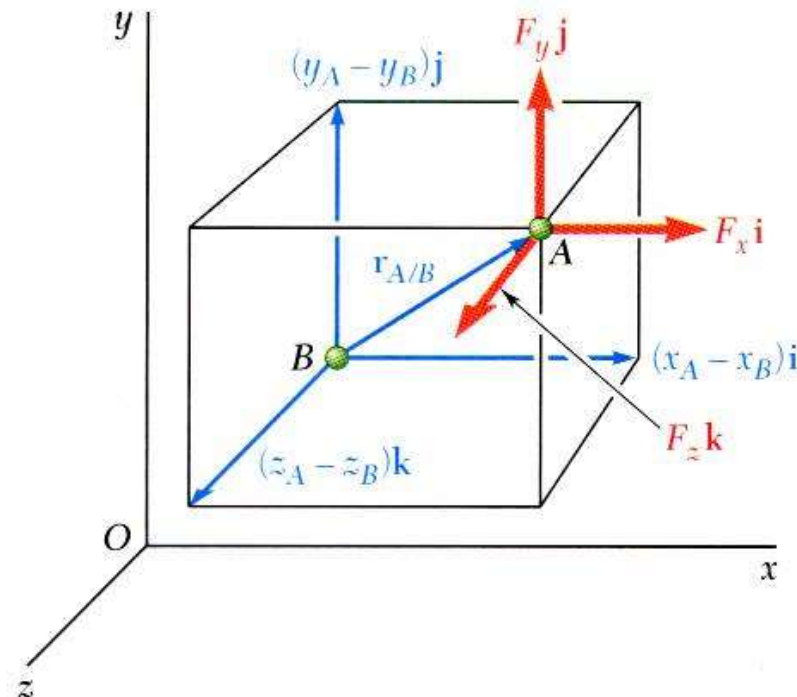
The moment of \mathbf{F} about B ,

$$\vec{M}_B = \vec{r}_{A/B} \times \vec{F}$$

$$\begin{aligned}\vec{r}_{A/B} &= \vec{r}_A - \vec{r}_B \\ &= (x_A - x_B)\vec{i} + (y_A - y_B)\vec{j} + (z_A - z_B)\vec{k}\end{aligned}$$

$$\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$$

$$\vec{M}_B = \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ (x_A - x_B) & (y_A - y_B) & (z_A - z_B) \\ F_x & F_y & F_z \end{pmatrix}$$



Rectangular Components of the Moment of a Force

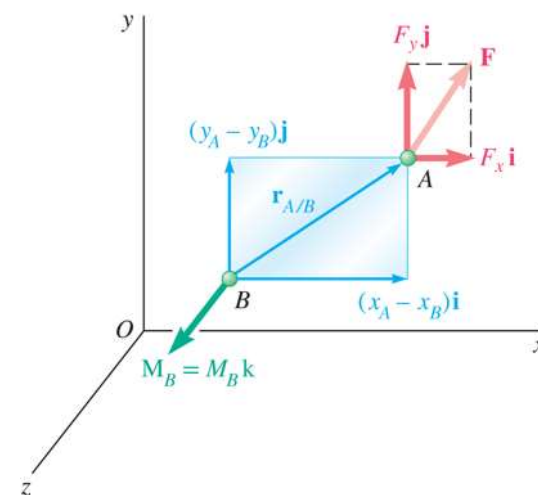
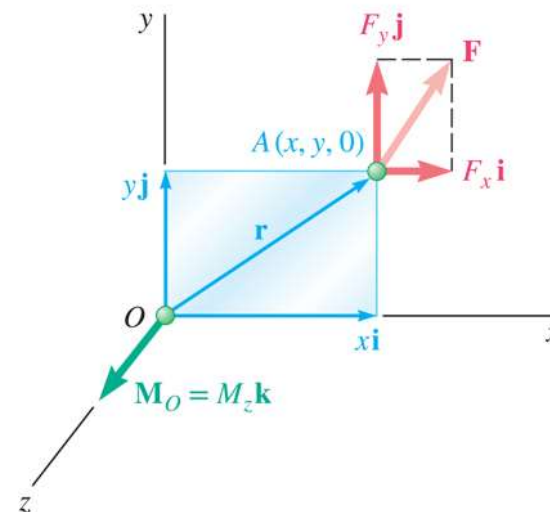
For two-dimensional structures,

$$\vec{M}_O = (xF_y - yF_z) \vec{k}$$

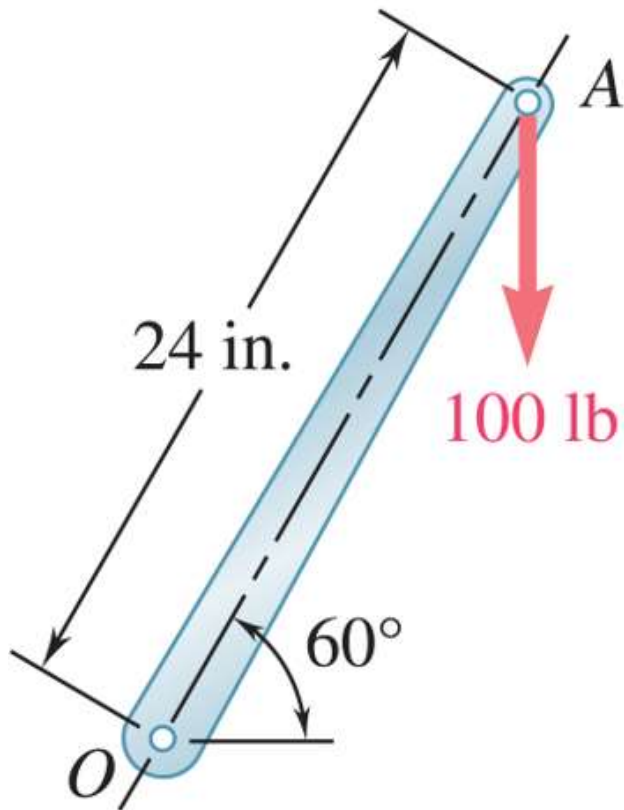
$$\begin{aligned} M_O &= M_Z \\ &= xF_y - yF_z \end{aligned}$$

$$\vec{M}_B = [(x_A - x_B)F_y - (y_A - y_B)F_z] \vec{k}$$

$$\begin{aligned} M_B &= M_Z \\ &= (x_A - x_B)F_y - (y_A - y_B)F_z \end{aligned}$$



Sample Problem 3.1

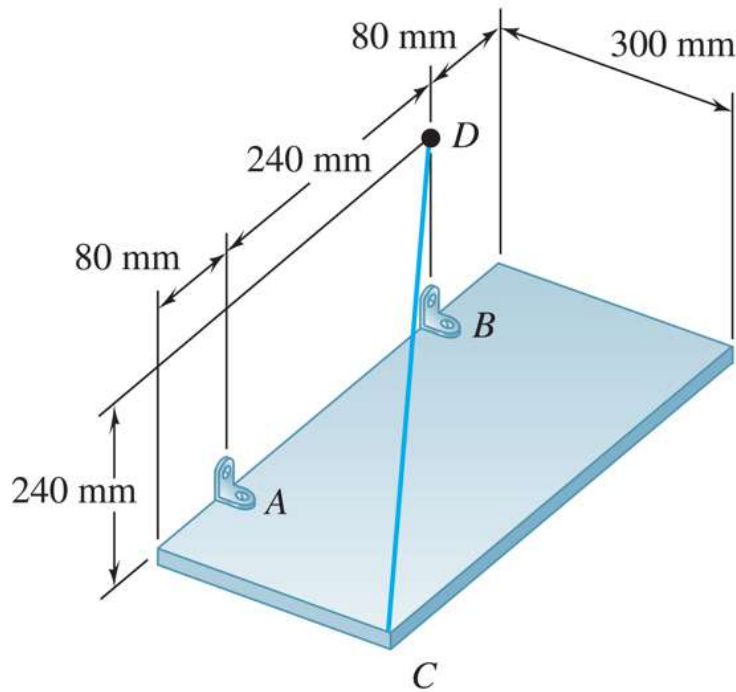


A 100-lb vertical force is applied to the end of a lever which is attached to a shaft (not shown) at O .

Determine:

- the moment about O ,
- the horizontal force at A which creates the same moment,
- the smallest force at A which produces the same moment,
- the location for a 240-lb vertical force to produce the same moment,
- whether any of the forces from b, c, and d is equivalent to the original force.

Sample Problem 3.4

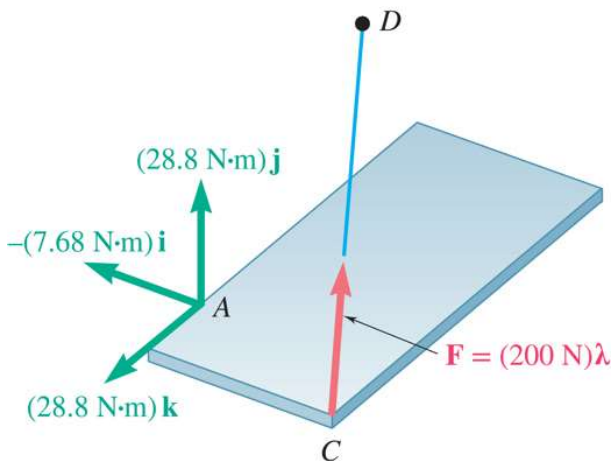


The rectangular plate is supported by the brackets at A and B and by a wire CD . Knowing that the tension in the wire is 200 N, determine the moment about A of the force exerted by the wire at C .

$$\vec{M}_A = \vec{r}_{C/A} \times \vec{F}$$

$$\vec{r}_{C/A} = \vec{r}_C - \vec{r}_A = (0.3 \text{ m})\vec{i} + (0.08 \text{ m})\vec{j}$$

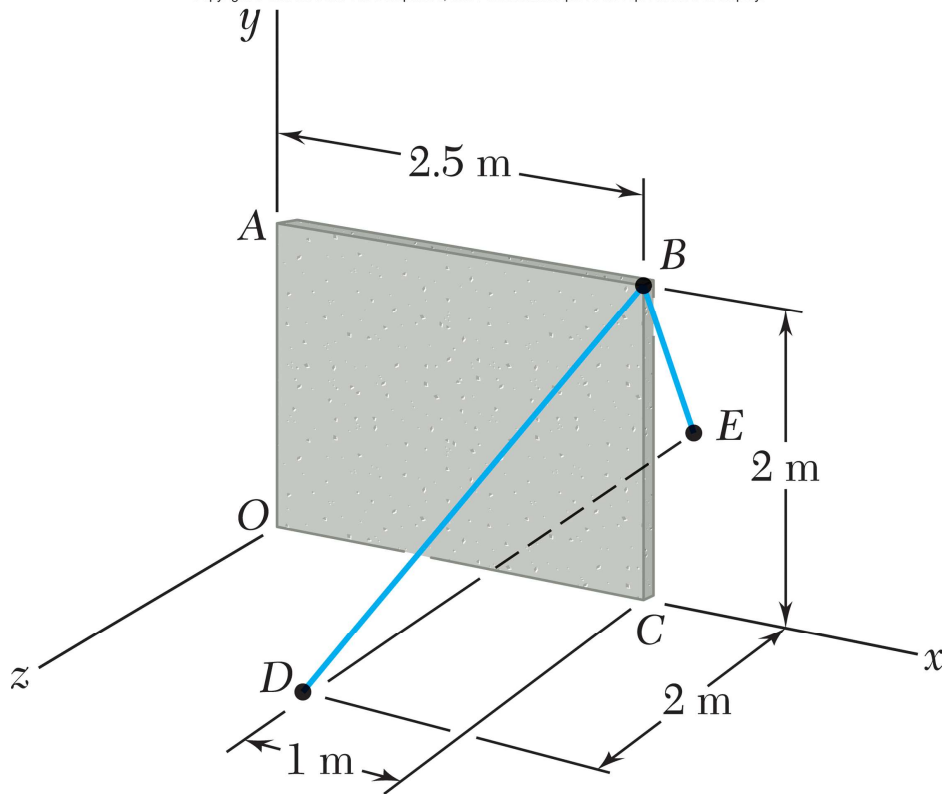
$$\begin{aligned} \vec{F} &= (200 \text{ N}) \frac{-(0.3 \text{ m})\vec{i} + (0.24 \text{ m})\vec{j} - (0.32 \text{ m})\vec{k}}{\sqrt{0.3^2 + 0.24^2 + 0.32^2} \text{ m}} \\ &= -(120 \text{ N})\vec{i} + (96 \text{ N})\vec{j} - (128 \text{ N})\vec{k} \end{aligned}$$



$$\vec{M}_A = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.3 & 0 & 0.08 \\ -120 & 96 & -128 \end{vmatrix} = -(7.68 \text{ N}\cdot\text{m})\vec{i} + (28.8 \text{ N}\cdot\text{m})\vec{j} + (28.8 \text{ N}\cdot\text{m})\vec{k}$$

Problem 3.24

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A precast concrete wall section is temporarily held by two cables as shown. Knowing that the tension in cable BD is 900 N , determine the moment about point O of the force exerted by the cable at B .

$$\mathbf{M}_O = \mathbf{r}_{B/O} \times \mathbf{F}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2.5 & 2 & 0 \\ -300 & -600 & 600 \end{vmatrix}$$

$$= 1200\mathbf{i} - 1500\mathbf{j} + (-1500 + 600)\mathbf{k}$$

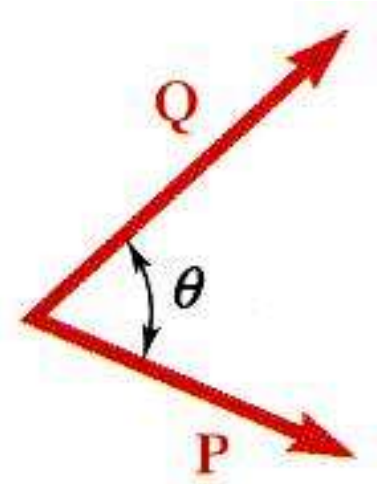
Scalar Product of Two Vectors

- The *scalar product* or *dot product* between two vectors \mathbf{P} and \mathbf{Q} is defined as

$$\vec{P} \bullet \vec{Q} = PQ \cos \theta \quad (\text{scalar result})$$

- Scalar products:

- are commutative, $\vec{P} \bullet \vec{Q} = \vec{Q} \bullet \vec{P}$
- are distributive, $\vec{P} \bullet (\vec{Q}_1 + \vec{Q}_2) = \vec{P} \bullet \vec{Q}_1 + \vec{P} \bullet \vec{Q}_2$
- are not associative, $(\vec{P} \bullet \vec{Q}) \bullet \vec{S} = \text{undefined}$



- Scalar products with Cartesian unit components,

$$\vec{P} \bullet \vec{Q} = (P_x \vec{i} + P_y \vec{j} + P_z \vec{k}) \bullet (Q_x \vec{i} + Q_y \vec{j} + Q_z \vec{k})$$

$$\vec{i} \bullet \vec{i} = 1 \quad \vec{j} \bullet \vec{j} = 1 \quad \vec{k} \bullet \vec{k} = 1 \quad \vec{i} \bullet \vec{j} = 0 \quad \vec{j} \bullet \vec{k} = 0 \quad \vec{k} \bullet \vec{i} = 0$$

$$\vec{P} \bullet \vec{Q} = P_x Q_x + P_y Q_y + P_z Q_z$$

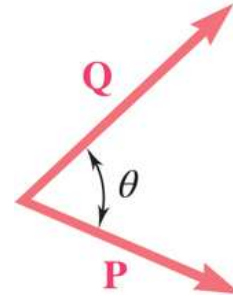
$$\vec{P} \bullet \vec{P} = P_x^2 + P_y^2 + P_z^2 = P^2$$

Applications of the Scalar Product

- Angle between two vectors:

$$\vec{P} \cdot \vec{Q} = PQ \cos \theta = P_x Q_x + P_y Q_y + P_z Q_z$$

$$\cos \theta = \frac{P_x Q_x + P_y Q_y + P_z Q_z}{PQ}$$



- Projection of a vector on a given axis:

$$P_{OL} = P \cos \theta = \text{projection of } P \text{ along } OL$$

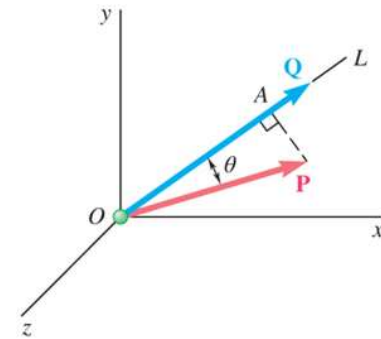
$$\vec{P} \cdot \vec{Q} = PQ \cos \theta$$

$$\frac{\vec{P} \cdot \vec{Q}}{Q} = P \cos \theta = P_{OL}$$

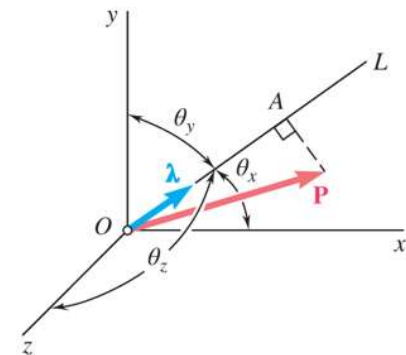
- For an axis defined by a unit vector:

$$P_{OL} = \vec{P} \cdot \vec{\lambda}$$

$$= P_x \cos \theta_x + P_y \cos \theta_y + P_z \cos \theta_z$$



(b)



(c)

Mixed Triple Product of Three Vectors

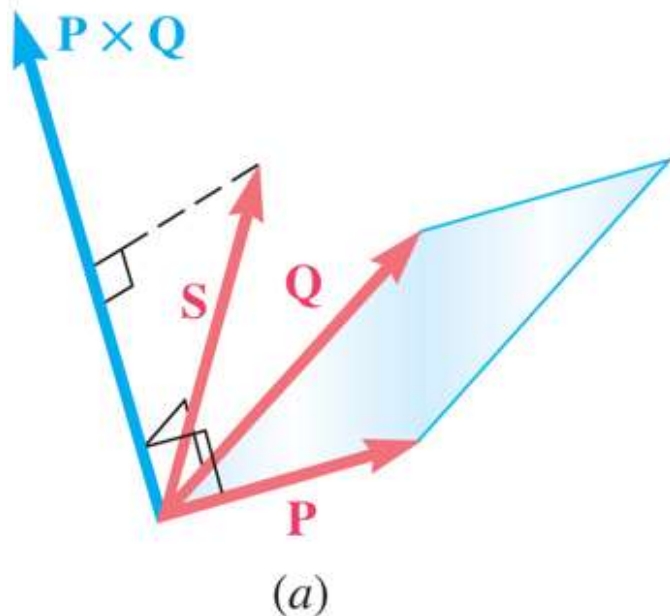
- Mixed triple product of three vectors,
 $\vec{S} \cdot (\vec{P} \times \vec{Q}) = \text{scalar result}$

- The six mixed triple products formed from \mathbf{S} , \mathbf{P} , and \mathbf{Q} have equal magnitudes but not the same sign,

$$\begin{aligned} \vec{S} \cdot (\vec{P} \times \vec{Q}) &= \vec{P} \cdot (\vec{Q} \times \vec{S}) = \vec{Q} \cdot (\vec{S} \times \vec{P}) \\ &= -\vec{S} \cdot (\vec{Q} \times \vec{P}) = -\vec{P} \cdot (\vec{S} \times \vec{Q}) = -\vec{Q} \cdot (\vec{P} \times \vec{S}) \end{aligned}$$

$$\begin{aligned} \vec{S} \cdot (\vec{P} \times \vec{Q}) &= S_x (P_y Q_z - P_z Q_y) + S_y (P_z Q_x - P_x Q_z) \\ &\quad + S_z (P_x Q_y - P_y Q_x) \end{aligned}$$

$$= \begin{pmatrix} S_x & S_y & S_z \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{pmatrix}$$



Chapter 3 Exercises

- A force of $F = 2 \mathbf{i} + 3 \mathbf{j} + 4 \mathbf{k}$ is applied to an object at a point $(3, -3, 2)$ from the origin. What is the moment exerted about the origin?
- To exert the same moment about the origin, what is the minimum force that is required to be applied at the same point of the object?

Moment of a Force About a Given Axis

- Moment \mathbf{M}_O of a force \mathbf{F} applied at the point A about a point O ,

$$\vec{M}_O = \vec{r} \times \vec{F}$$

- Scalar moment M_{OL} about an axis OL is the projection of the moment vector \mathbf{M}_O onto the axis,

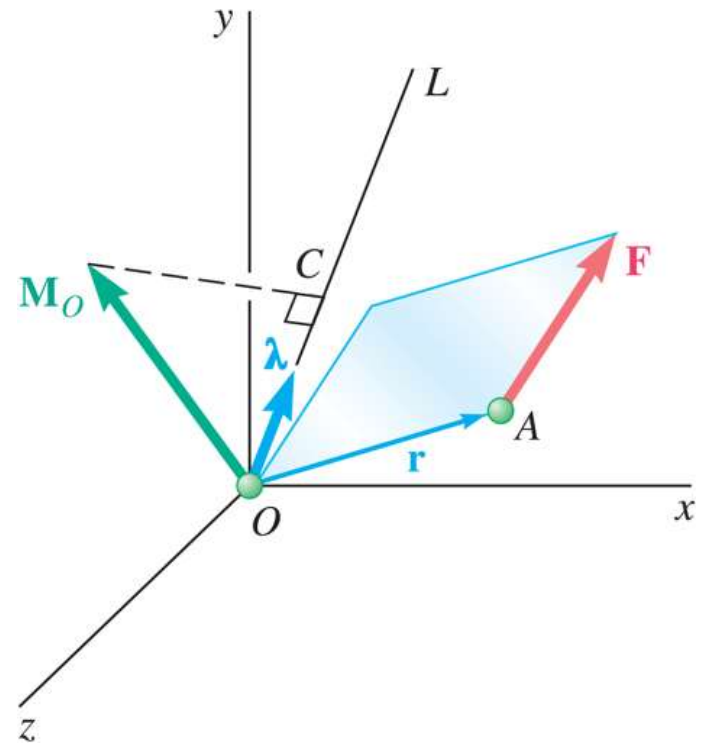
$$M_{OL} = \vec{\lambda} \cdot \vec{M}_O = \vec{\lambda} \cdot (\vec{r} \times \vec{F})$$

- Moments of \mathbf{F} about the coordinate axes,

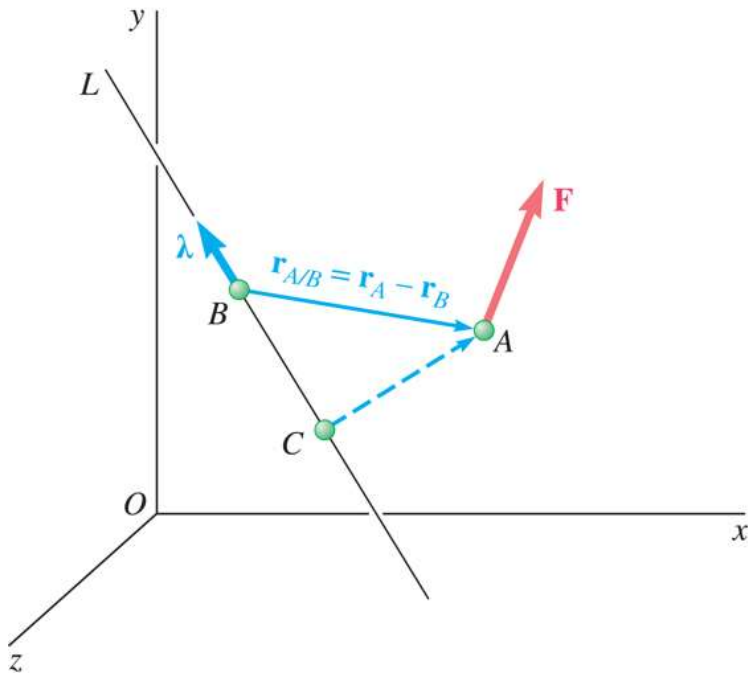
$$M_x = yF_z - zF_y$$

$$M_y = zF_x - xF_z$$

$$M_z = xF_y - yF_x$$



Moment of a Force About a Given Axis



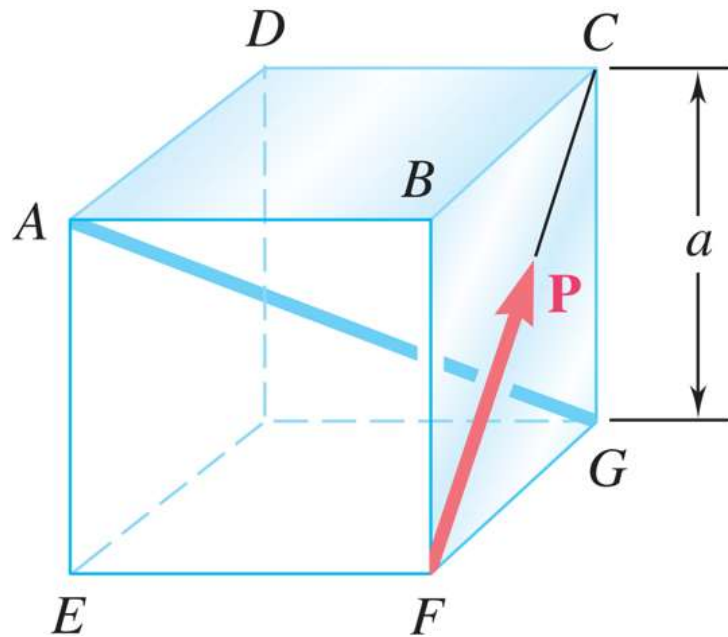
- Moment of a force about an arbitrary axis,

$$\begin{aligned}M_{BL} &= \vec{\lambda} \cdot \vec{M}_B \\ &= \vec{\lambda} \cdot (\vec{r}_{A/B} \times \vec{F})\end{aligned}$$

$$\vec{r}_{A/B} = \vec{r}_A - \vec{r}_B$$

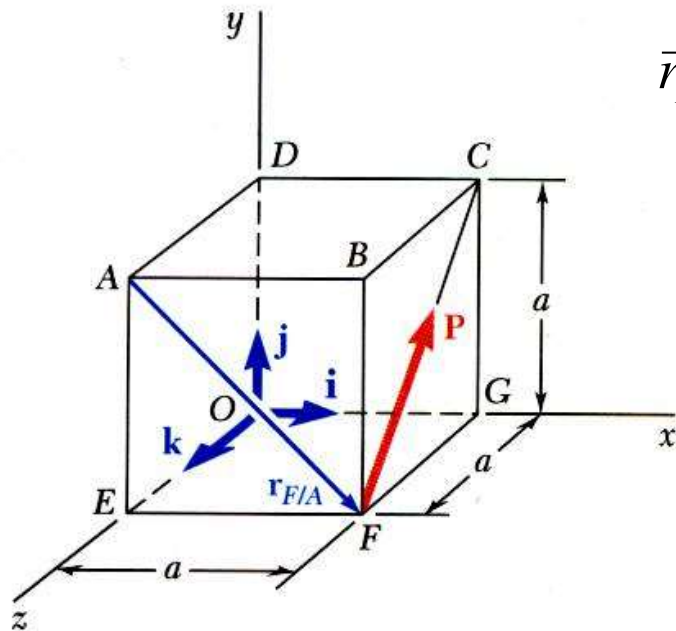
- The result is independent of the point B along the given axis. For example, the same result can be obtained using $\vec{r}_{A/C}$

Sample Problem 3.5



A cube is acted on by a force \mathbf{P} as shown. Determine the moment of \mathbf{P}

- about A $\vec{M}_A = \vec{r}_{F/A} \times \vec{P}$
- about the edge AB and
- about the diagonal AG of the cube.
- Determine the perpendicular distance between AG and FC .



$$\vec{r}_{F/A} = a\vec{i} - a\vec{j} = a(\vec{i} - \vec{j}) \quad \vec{P} = P(\sqrt{2}\vec{i} + \sqrt{2}\vec{j}) = P\sqrt{2}(\vec{i} + \vec{j})$$

$$\vec{M}_A = (aP\sqrt{2})(\vec{i} + \vec{j} + \vec{k})$$

$$M_{AB} = \vec{i} \cdot \vec{M}_A = aP\sqrt{2}$$

$$M_{AG} = \vec{\lambda}_{AG} \cdot \vec{M}_A \quad \vec{\lambda}_{AG} = \frac{1}{\sqrt{3}}(\vec{i} - \vec{j} - \vec{k})$$

$$M_{AG} = -\frac{aP}{\sqrt{6}}$$

$$d = \frac{a}{\sqrt{6}}$$

Moment of a Couple

- Two forces F and $-F$ having the same magnitude, parallel lines of action, and opposite sense are said to form a *couple*.

- Moment of the couple,

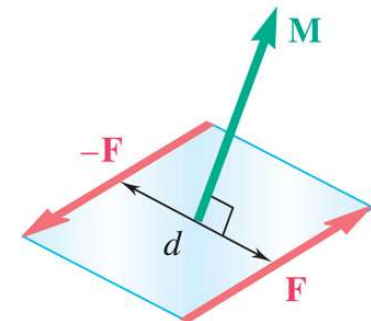
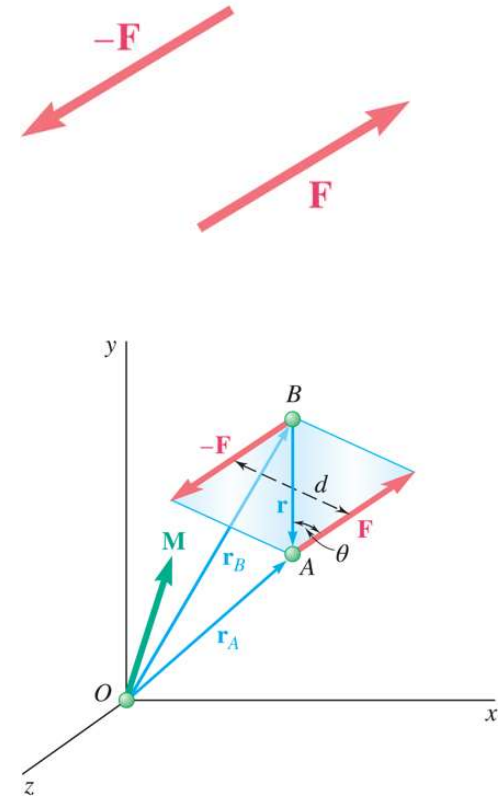
$$\vec{M} = \vec{r}_A \times \vec{F} + \vec{r}_B \times (-\vec{F})$$

$$= (\vec{r}_A - \vec{r}_B) \times \vec{F}$$

$$= \vec{r} \times \vec{F}$$

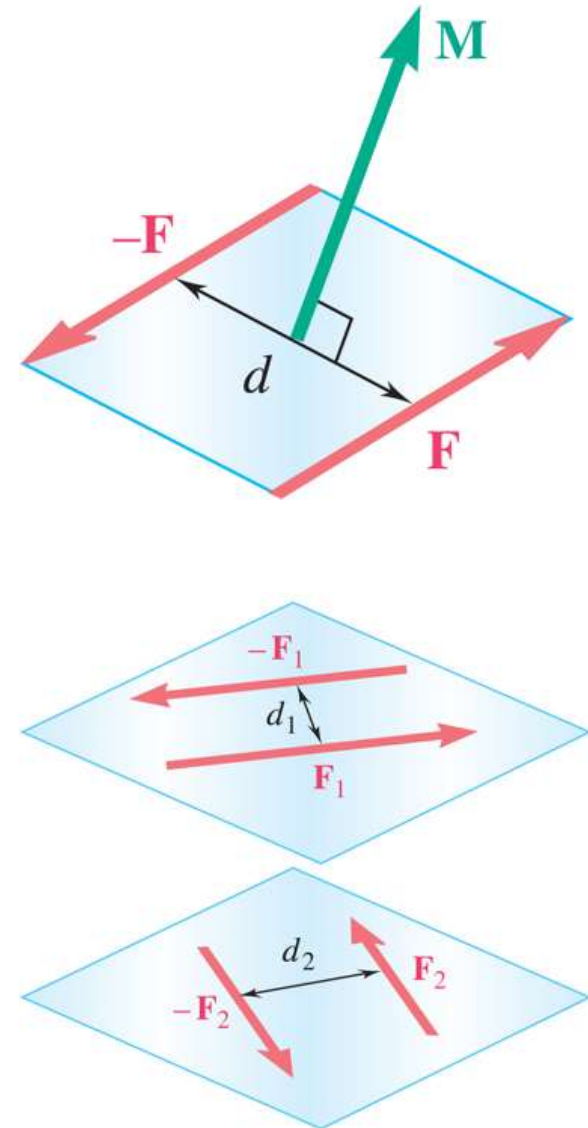
$$M = rF \sin \theta = Fd$$

- The moment vector of the couple is independent of the choice of the origin of the coordinate axes, i.e., it is a *free vector* that can be applied at any point with the same effect.

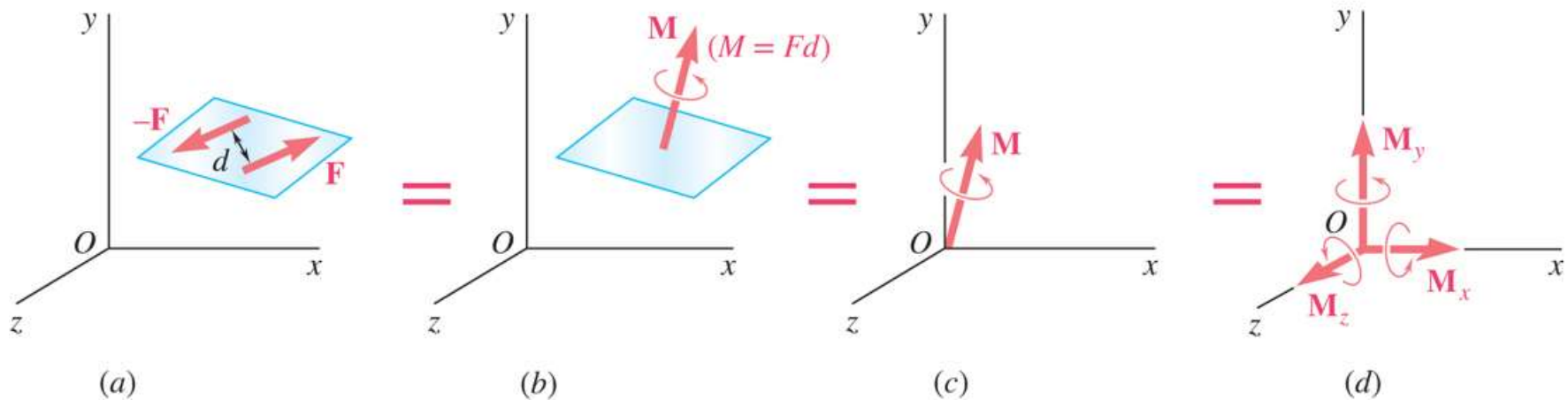


Moment of a Couple

- Two couples will have equal moments if.
- $F_1d_1 = F_2d_2$.
- the two couples lie in parallel planes, and.
- the two couples have the same sense or the tendency to cause rotation in the same direction.

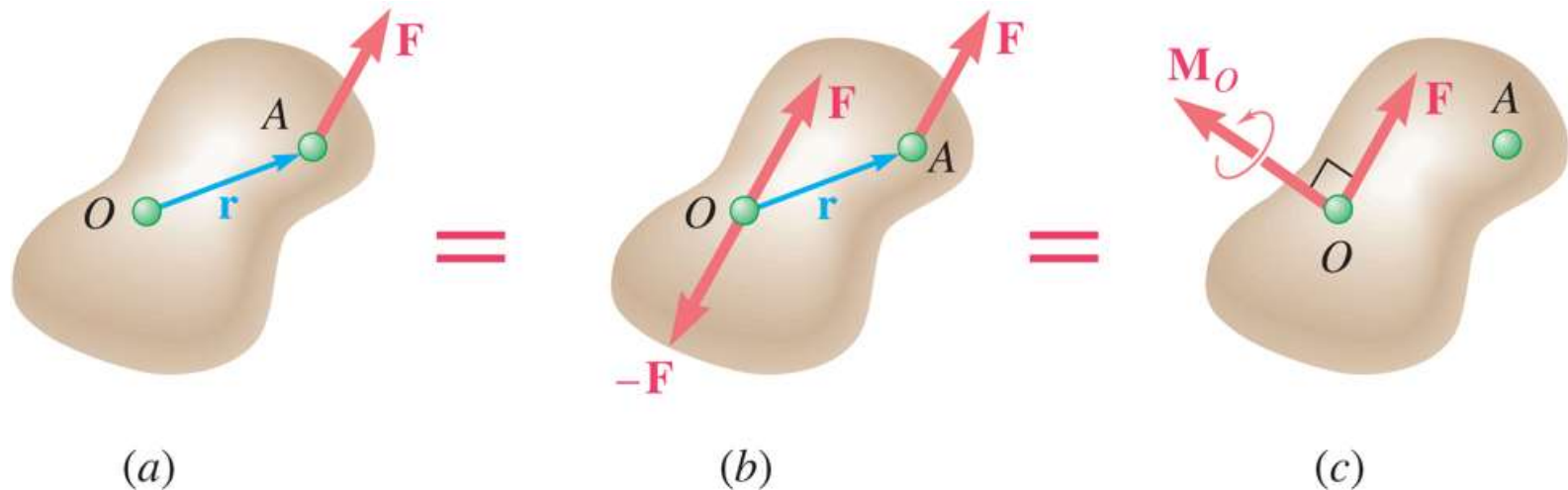


Couples Can Be Represented by Vectors



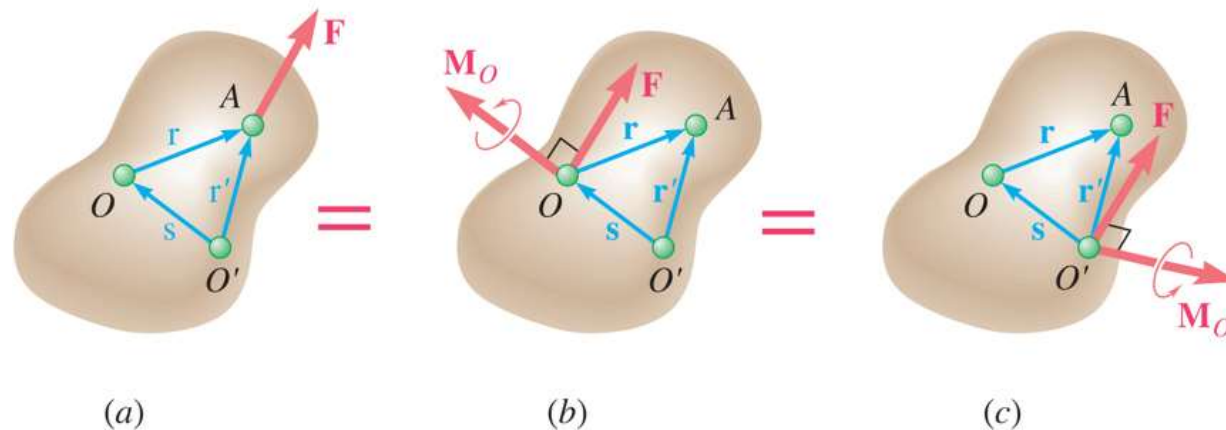
- A couple can be represented by a vector with magnitude and direction equal to the moment of the couple.
- *Couple vectors* obey the law of addition of vectors.
- Couple vectors are free vectors, i.e., there is no point of application – it simply acts on the body.
- Couple vectors may be resolved into component vectors.

Resolution of a Force Into a Force at O and a Couple



- Force vector F can not be simply moved to O without modifying its action on the body. Attaching equal and opposite force vectors at O produces no net effect on the body. The three forces may be replaced by an equivalent force vector and couple vector, i.e, a *force-couple system*.

Resolution of a Force Into a Force at O and a Couple



- Moving F from A to a different point O' requires the addition of a different couple vector $M_{O'}$,

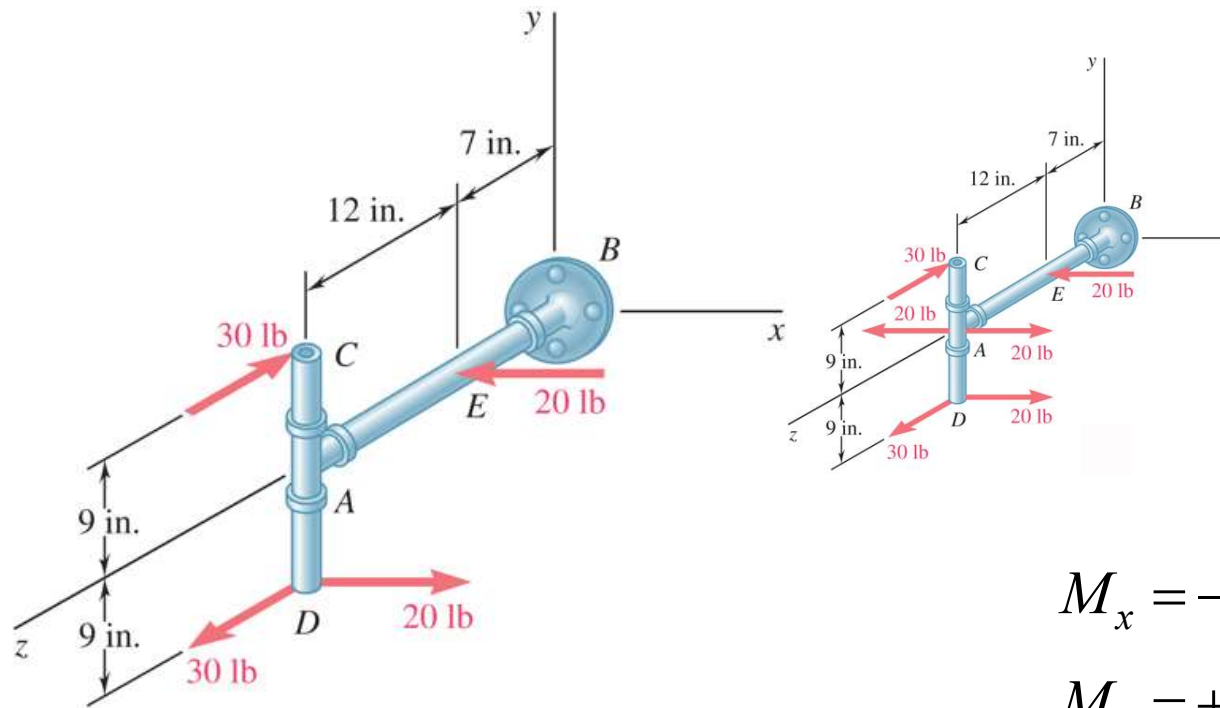
$$\vec{M}_{O'} = \vec{r}' \times \vec{F}$$

- The moments of F about O and O' are related,

$$\begin{aligned} \vec{M}_{O'} &= \vec{r}' \times \vec{F} = (\vec{r} + \vec{s}) \times \vec{F} = \vec{r} \times \vec{F} + \vec{s} \times \vec{F} \\ &= \vec{M}_O + \vec{s} \times \vec{F} \end{aligned}$$

- Moving the force-couple system from O to O' requires the addition of the moment of the force at O about O' .

Sample Problem 3.6



Attach equal and opposite 20 lb forces in the $\pm x$ direction at A

Determine the components of the single couple equivalent to the couples shown.

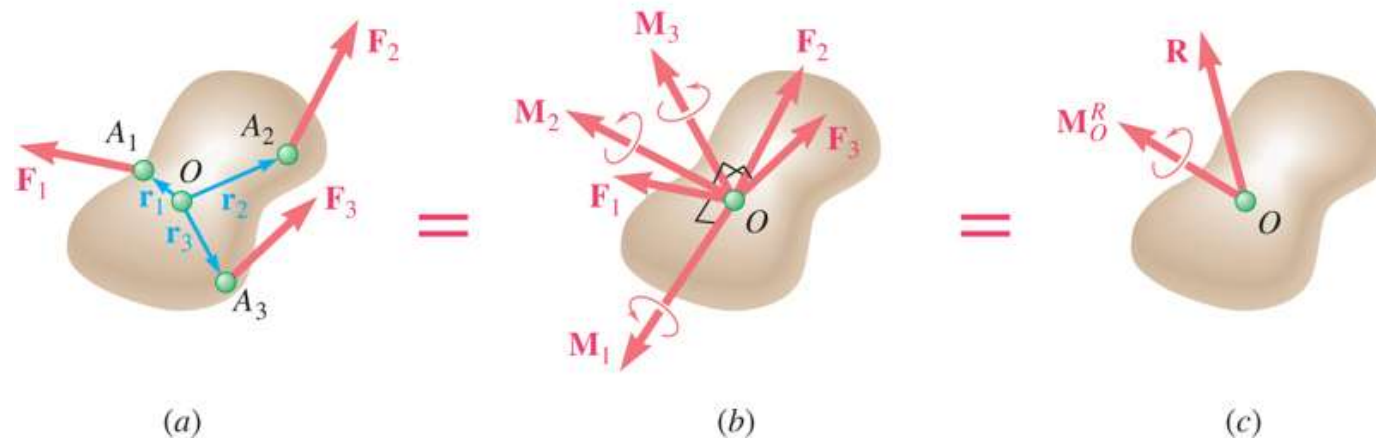
$$M_x = -(30 \text{ lb})(18 \text{ in.}) = -540 \text{ lb} \cdot \text{in.}$$

$$M_y = +(20 \text{ lb})(12 \text{ in.}) = +240 \text{ lb} \cdot \text{in.}$$

$$M_z = +(20 \text{ lb})(9 \text{ in.}) = +180 \text{ lb} \cdot \text{in.}$$

$$\vec{M} = -(540 \text{ lb} \cdot \text{in.}) \vec{i} + (240 \text{ lb} \cdot \text{in.}) \vec{j} + (180 \text{ lb} \cdot \text{in.}) \vec{k}$$

System of Forces: Reduction to a Force and Couple



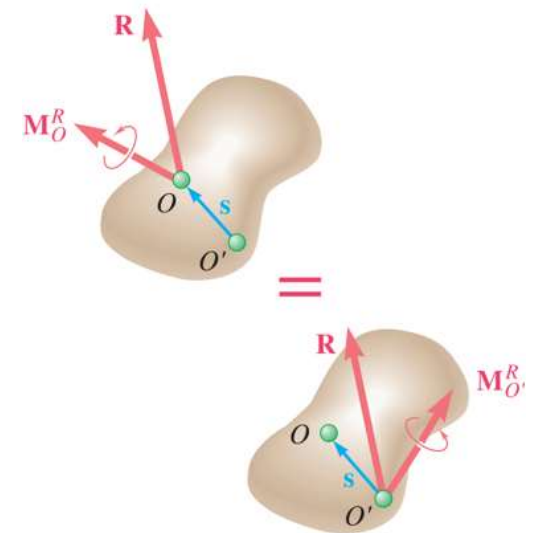
- The force and couple vectors may be combined into a resultant force vector and a resultant couple vector,

$$\vec{R} = \sum \vec{F} \quad \vec{M}_O^R = \sum (\vec{r} \times \vec{F})$$

- The force-couple system at O may be moved to O' with the addition of the moment of \mathbf{R} about O' ,

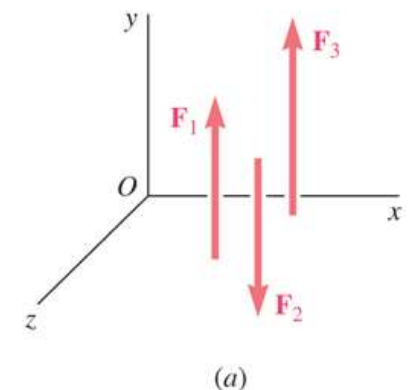
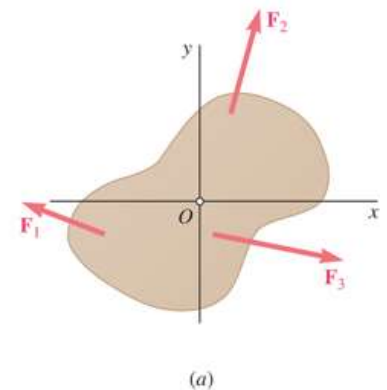
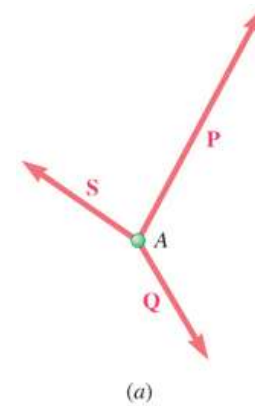
$$\vec{M}_{O'n}^R = \vec{M}_O^R + \vec{s} \times \vec{R}$$

- Two systems of forces are equivalent if they can be reduced to the same force-couple system.

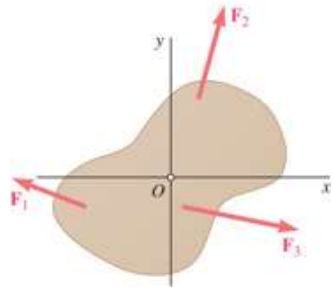


Further Reduction of a System of Forces

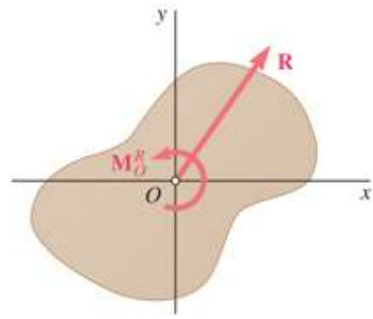
- If the resultant force and couple at O are mutually perpendicular, they can be replaced by a single force acting along a new line of action.
- The resultant force-couple system for a system of forces will be mutually perpendicular if:
 - 1) the forces are concurrent,
 - 2) the forces are coplanar, or
 - 3) the forces are parallel.



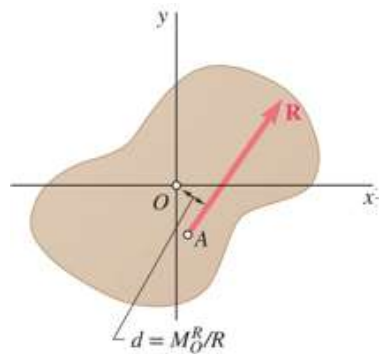
Further Reduction of a System of Forces



(a)



(b)



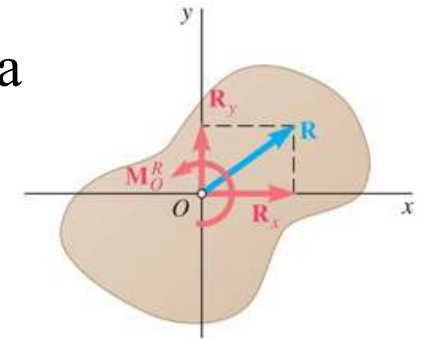
(c)

- System of coplanar forces is reduced to a force-couple system \vec{R} and \vec{M}_O^R that is mutually perpendicular.

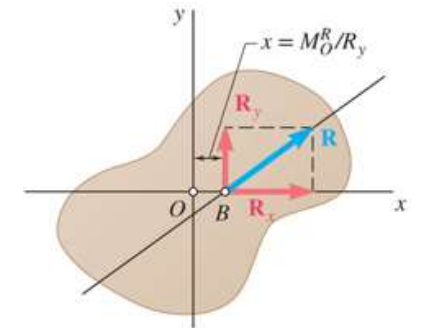
- System can be reduced to a single force by moving the line of action of \vec{R} until its moment about O becomes \vec{M}_O^R

- In terms of rectangular coordinates,

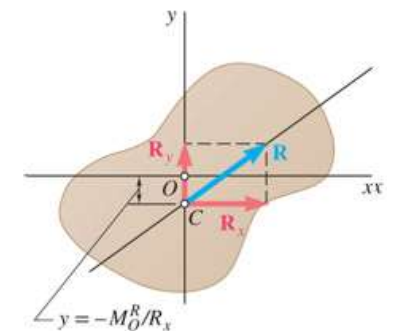
$$xR_y - yR_x = M_O^R$$



(a)

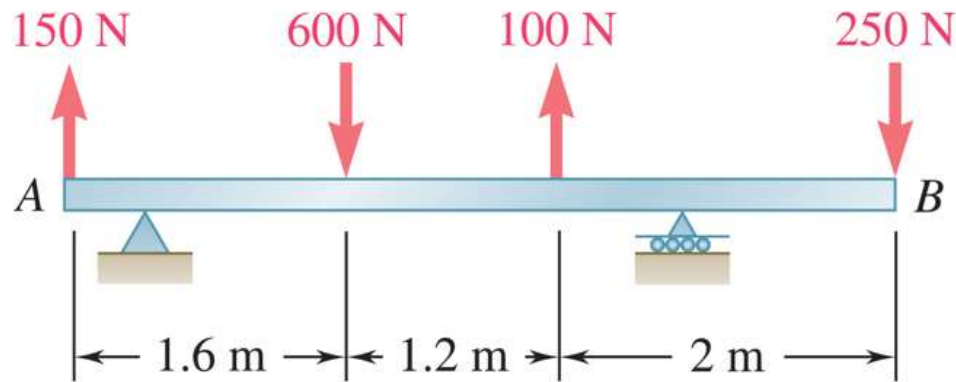


(b)



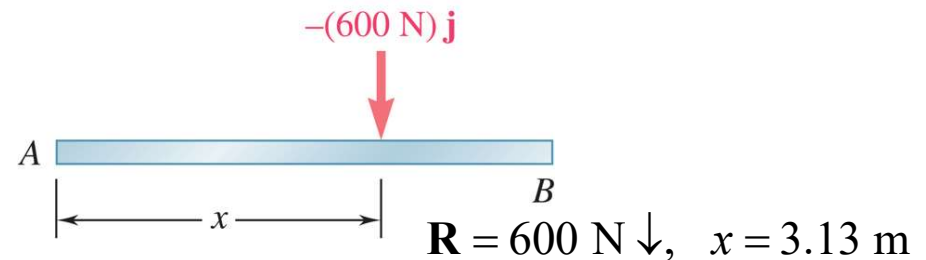
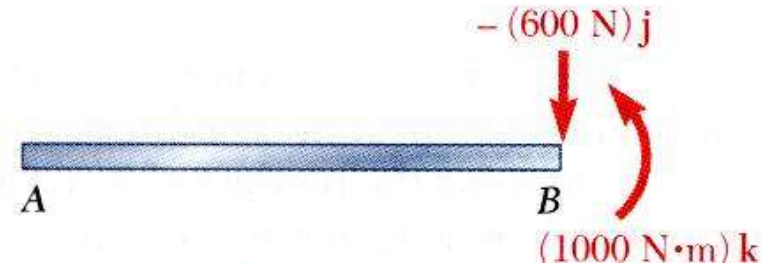
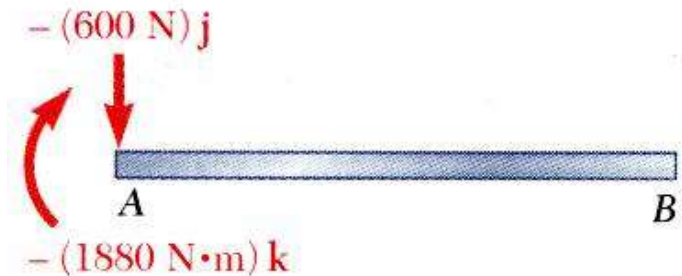
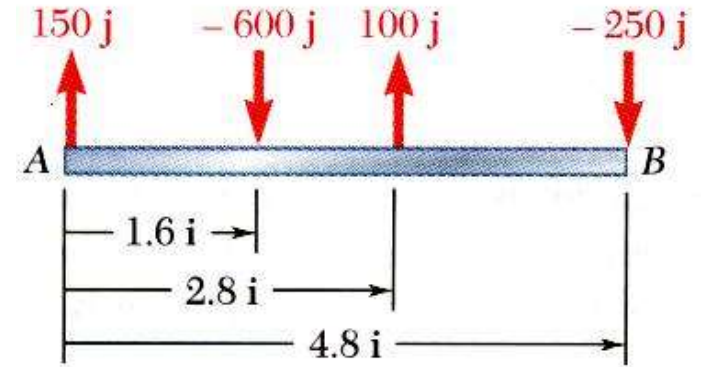
(c)

Sample Problem 3.8

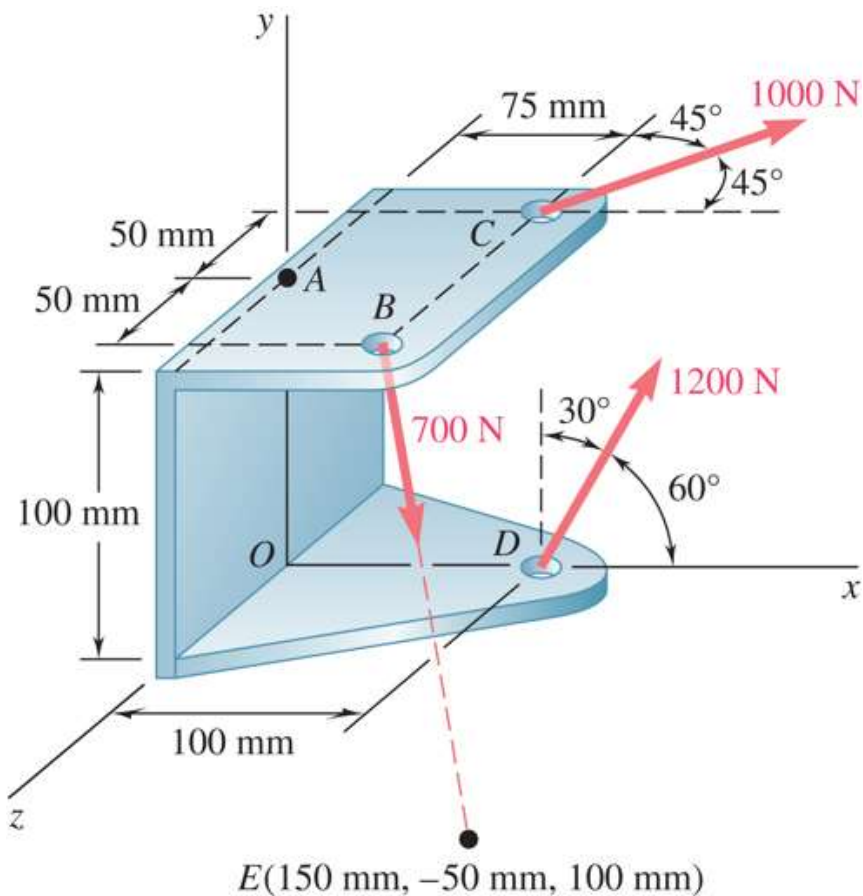


For the beam, reduce the system of forces shown to (a) an equivalent force-couple system at A , (b) an equivalent force couple system at B , and (c) a single force or resultant.

Note: Since the support reactions are not included, the given system will not maintain the beam in equilibrium.



Sample Problem 3.10



Three cables are attached to the bracket as shown. Replace the forces with an equivalent force-couple system at A .

Solution by brute force:

$$\vec{r}_{B/A} = 0.075\vec{i} + 0.050\vec{k} \text{ (m)}$$

$$\vec{r}_{C/A} = 0.075\vec{i} - 0.050\vec{k} \text{ (m)}$$

$$\vec{r}_{D/A} = 0.100\vec{i} - 0.100\vec{j} \text{ (m)}$$

$$\vec{F}_B = 300\vec{i} - 600\vec{j} + 200\vec{k} \text{ (N)}$$

$$\vec{F}_C = 707\vec{i} - 707\vec{j} \text{ (N)}$$

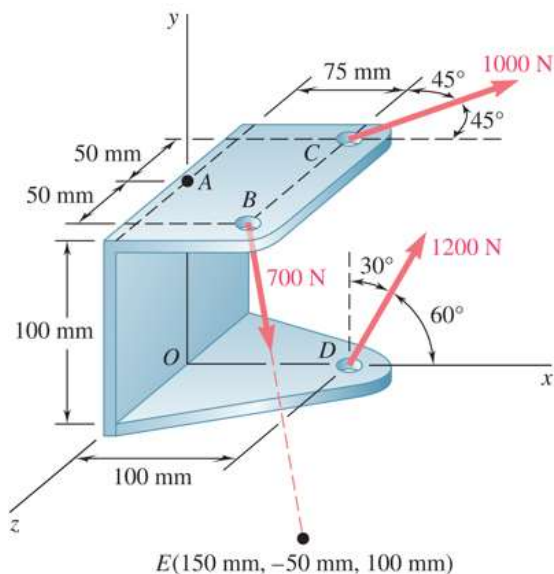
$$\vec{F}_D = 600\vec{i} + 1039\vec{j} \text{ (N)}$$

Sample Problem 3.10

- Compute the equivalent force,

$$\begin{aligned}\vec{R} &= \sum \vec{F} \\ &= (300 + 707 + 600)\vec{i} \\ &\quad + (-600 + 1039)\vec{j} \\ &\quad + (200 - 707)\vec{k}\end{aligned}$$

$$\vec{R} = 1607\vec{i} + 439\vec{j} - 507\vec{k} \text{ (N)}$$



- Compute the equivalent couple,

$$\vec{M}_A^R = \sum (\vec{r} \times \vec{F})$$

$$\vec{r}_{B/A} \times \vec{F}_B = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.075 & 0 & 0.050 \\ 300 & -600 & 200 \end{vmatrix} = 30\vec{i} - 45\vec{k}$$

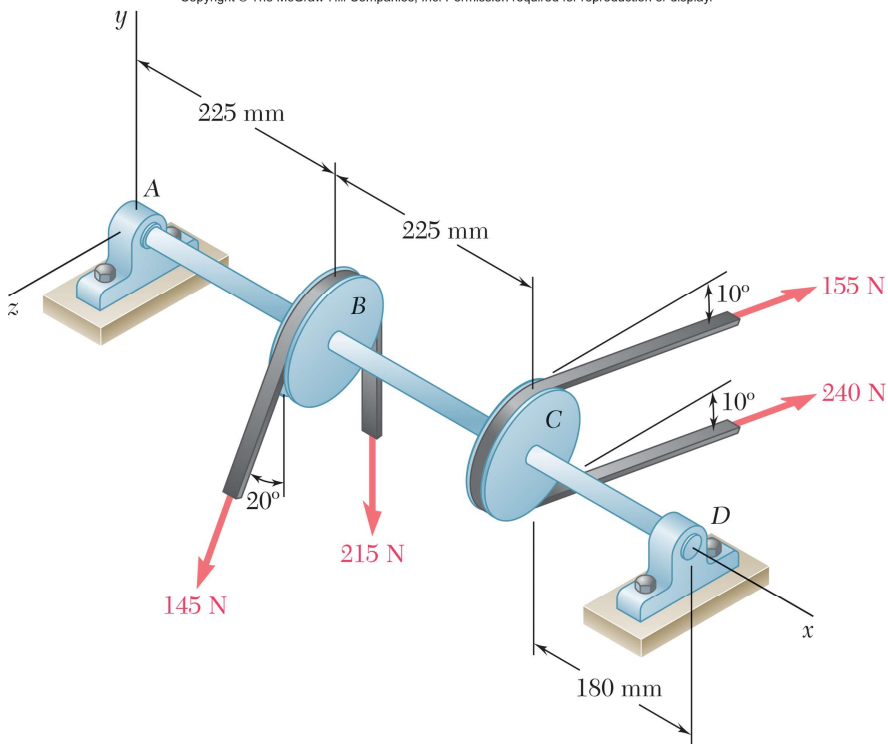
$$\vec{r}_{C/A} \times \vec{F}_C = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.075 & 0 & -0.050 \\ 707 & 0 & -707 \end{vmatrix} = 17.68\vec{j}$$

$$\vec{r}_{D/A} \times \vec{F}_D = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.100 & -0.100 & 0 \\ 600 & 1039 & 0 \end{vmatrix} = 163.9\vec{k}$$

$$\vec{M}_A^R = 30\vec{i} + 17.68\vec{j} + 118.9\vec{k}$$

Problem 3.120

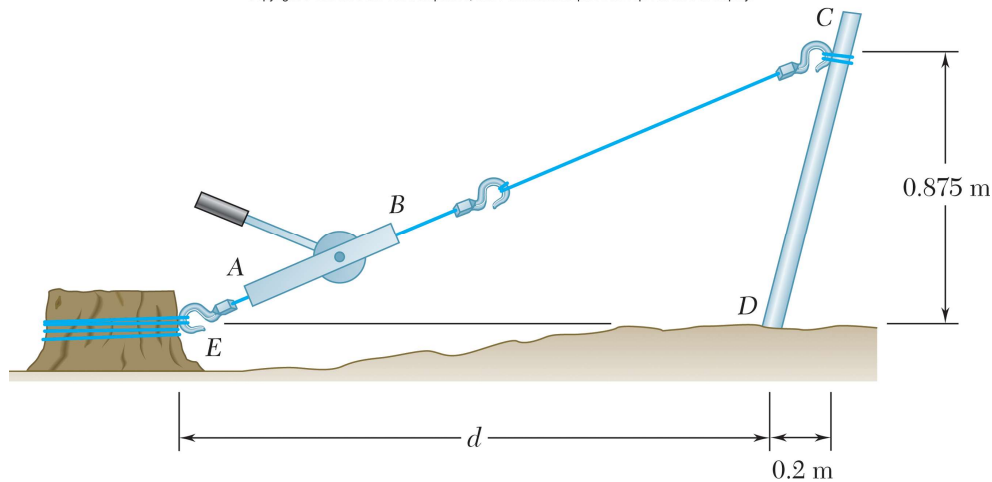
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Two 150-mm-diameter pulleys are mounted on line shaft AD . The belts at B and C lie in vertical planes parallel to the yz plane. Replace the belt forces shown with an equivalent force-couple system at A .

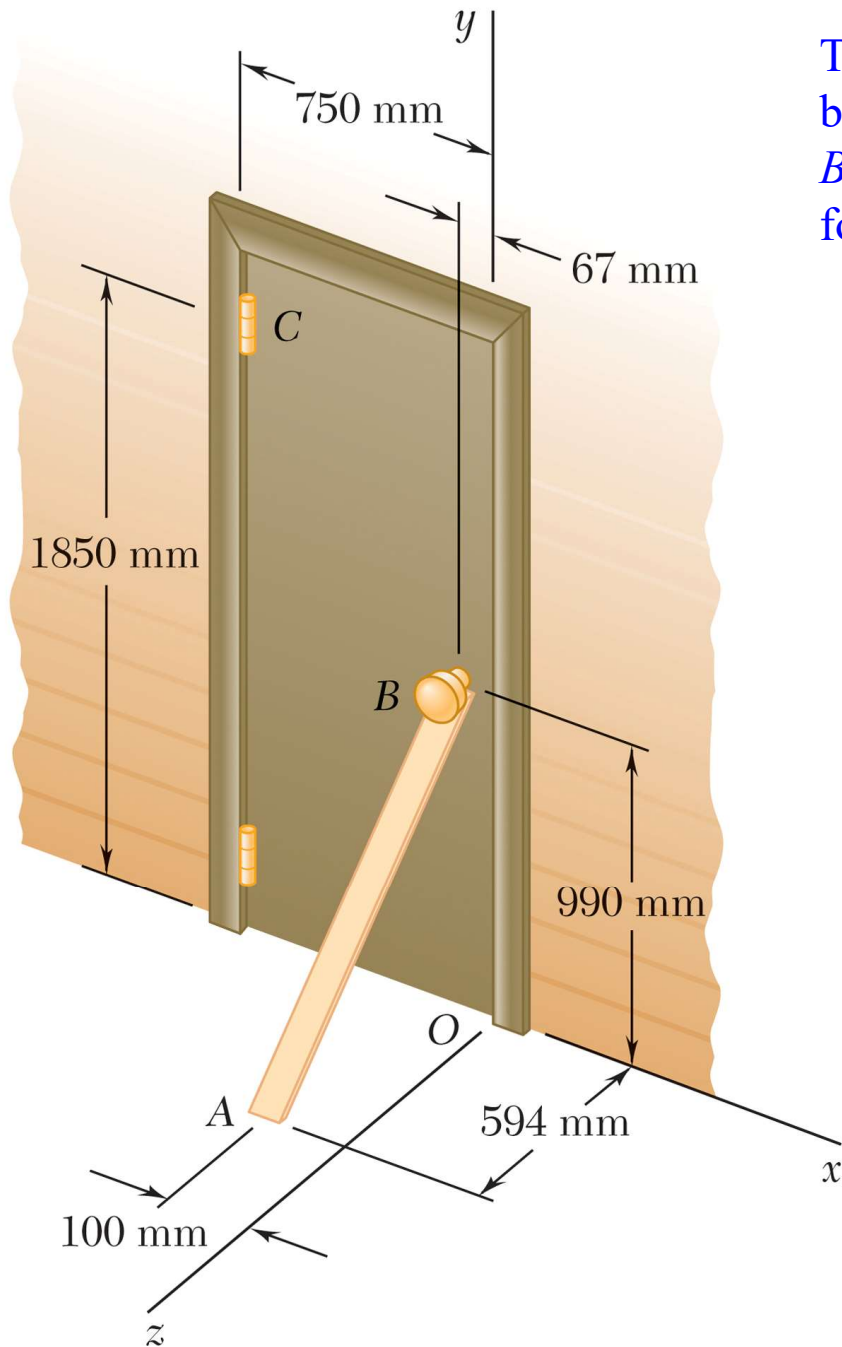
Problem 3.13

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It is known that a force with a moment of $960\text{ N}\cdot\text{m}$ about D is required to straighten the fence post CD . If the capacity of winch puller AB is 2400 N , determine the minimum value of distance d to create the specified moment about point D .

Problem 3.97



To keep a door closed, a wooden stick is wedged between the floor and the doorknob. The stick exerts at B a 175-N force directed along line AB . Replace that force with an equivalent force-couple system at C .