Ch. 5 Distributed Forces: Centroids and CG

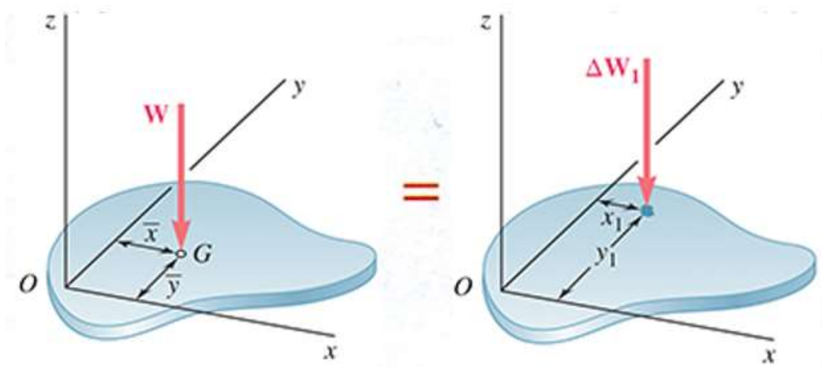
- The earth exerts a gravitational force on each of the particles forming a body. These forces can be replaced by a single equivalent force equal to the weight of the body and applied at the *center of gravity* for the body.

- The *centroid of an area* is analogous to the center of gravity of a body; it is the "center of area." The concept of the *first moment of an area* is used to locate the centroid.

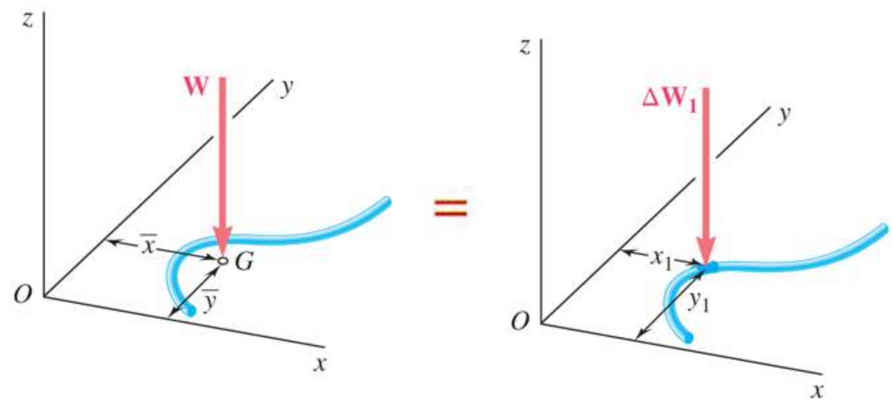
- Determination of the area of a *surface of revolution* and the volume of a *body of revolution* are accomplished with the *Theorems of Pappus-Guldinus*.

Center of Gravity of a 2D Body

- Center of gravity of a plate



- Center of gravity of a wire



$$\sum M_y \quad \bar{x}W = \sum x\Delta W$$

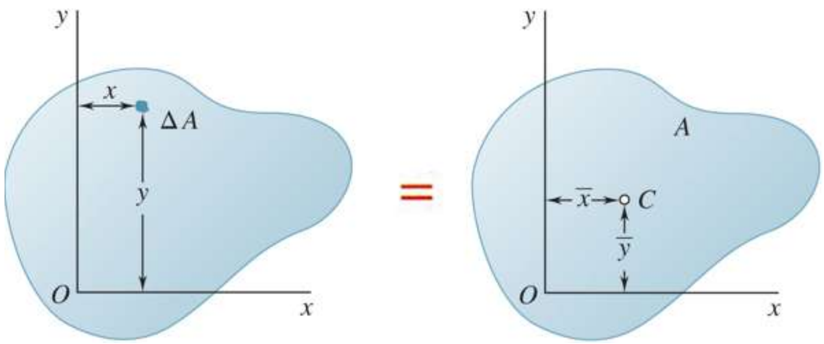
$$= \int x\, dW$$

$$\sum M_y \quad \bar{y}W = \sum y\Delta W$$

$$= \int y\, dW$$

Centroids and First Moments of Areas and Lines

- Centroid of an area



$$\bar{x}W = \int x\,dW$$

$$\bar{x}\left(\gamma A t\right) = \int x\left(\gamma t\right) dA$$

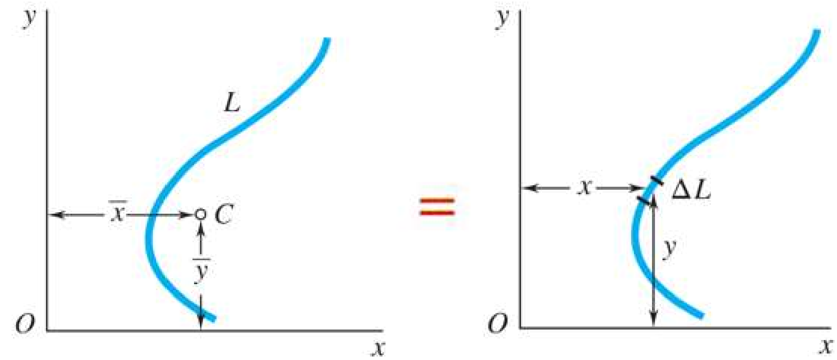
first moment (w.r.t x-axis)

$$\bar{x}A = \int x\,dA = Q_y$$

first moment (w.r.t y-axis)

$$\bar{y}A = \int y\,dA = Q_x$$

- Centroid of a line



γ: specific weight

***t*: thickness**

***a*: cross-section**

$$\bar{x}W = \int x\,dW$$

$$\bar{x}\left(\gamma L a\right) = \int x\left(\gamma a\right) dL$$

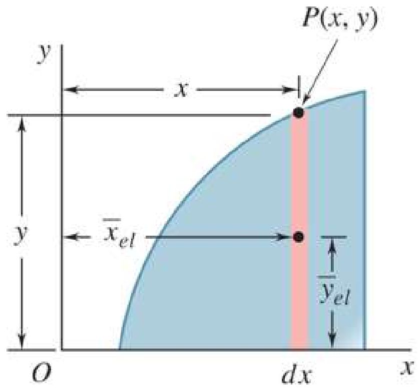
$$\bar{x}L = \int x\,dL$$

$$\bar{y}L = \int y\,dL$$

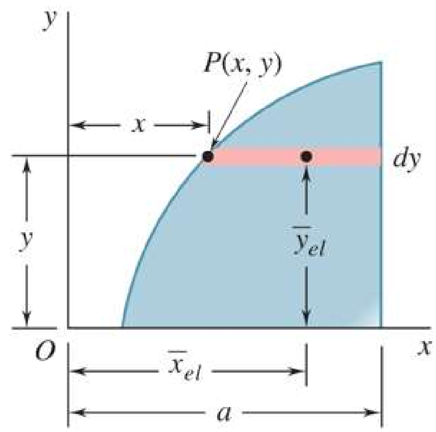
Determination of Centroids by Integration

$$\bar{x}A = \int x\,dA = \iint x\,dx\,dy = \int \bar{x}_{el}\,dA$$
$$\bar{y}A = \int y\,dA = \iint y\,dx\,dy = \int \bar{y}_{el}\,dA$$

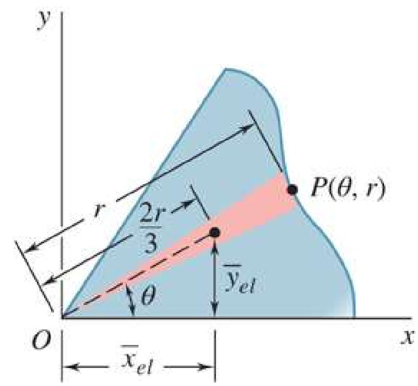
- Double integration to find the first moment may be avoided by defining *dA* as a thin rectangle or strip.



$$\bar{x}A = \int \bar{x}_{el}\,dA = \int x(y\,dx)$$
$$\bar{y}A = \int \bar{y}_{el}\,dA = \int \frac{y}{2}(y\,dx)$$

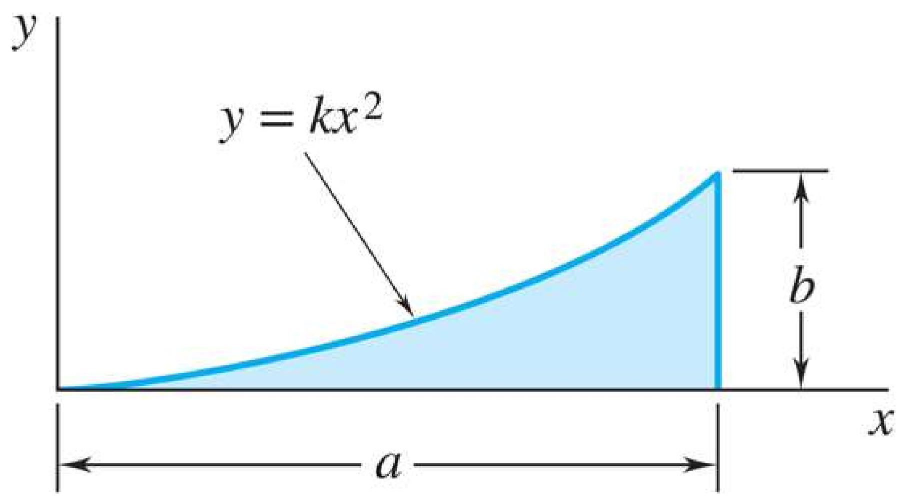


$$\bar{x}A = \int \bar{x}_{el}\,dA = \int \frac{a+x}{2}\left[(a-x)\,dx\right]$$
$$\bar{y}A = \int \bar{y}_{el}\,dA = \int y\left[(a-x)\,dx\right]$$



$$\bar{x}A = \int \bar{x}_{el}\,dA = \int \frac{2r}{3}\cos\theta\left(\frac{1}{2}r^2\,d\theta\right)$$
$$\bar{y}A = \int \bar{y}_{el}\,dA = \int \frac{2r}{3}\sin\theta\left(\frac{1}{2}r^2\,d\theta\right)$$

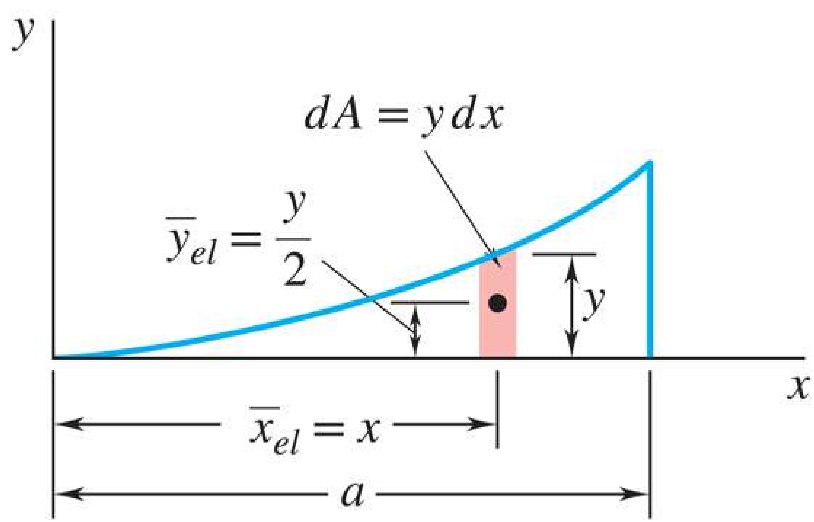
Sample Problem 5.4



Determine by direct integration the location of the centroid of a parabolic spandrel.

$$k = \frac{b}{a^2}$$

$$y = \frac{b}{a^2}x^2 \quad or \quad x = \frac{a}{b^{1/2}}y^{1/2}$$



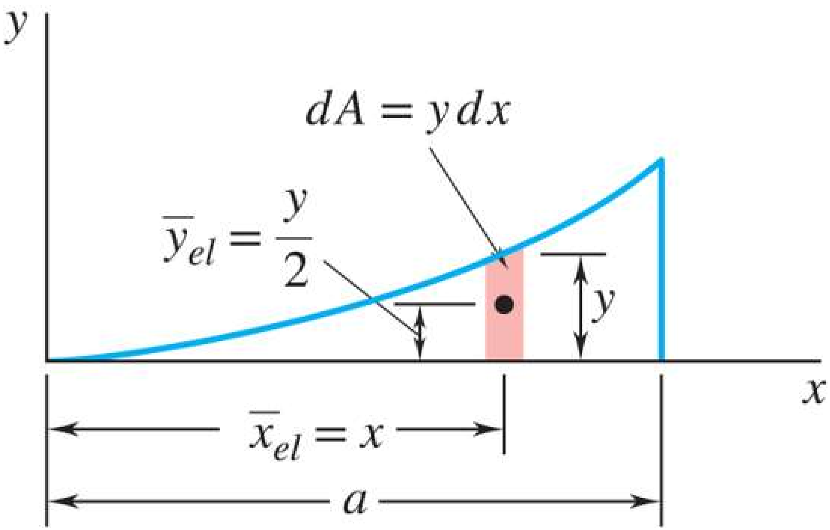
- Evaluate the total area.

$$A = \int dA$$

$$= \int y\,dx = \int_0^a \frac{b}{a^2}x^2 dx = \left[\frac{b}{a^2}\frac{x^3}{3}\right]_0^a$$

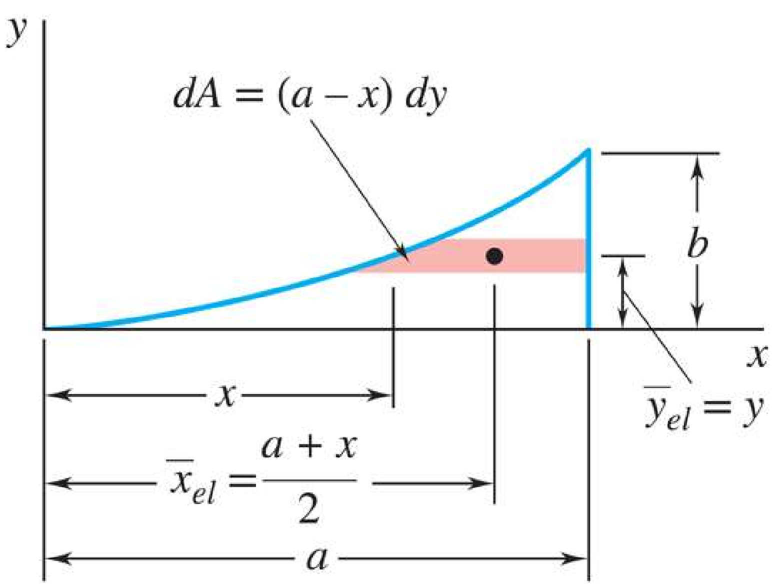
$$= \frac{ab}{3}$$

Sample Problem 5.4



$$Q_y = \int \bar{x}_{el}\,dA = \int xy\,dx = \int_0^a x\left(\frac{b}{a^2}x^2\right)dx = \frac{a^2 b}{4}$$

$$Q_x = \int \bar{y}_{el}\,dA = \int \frac{y}{2}\,y\,dx = \int_0^a \frac{1}{2}\left(\frac{b}{a^2}x^2\right)^2 dx = \frac{ab^2}{10}$$



$$Q_y = \int \bar{x}_{el}\,dA = \int \frac{a+x}{2}(a-x)\,dy = \int_0^b \frac{a^2 - x^2}{2}\,dy$$

$$= \frac{1}{2}\int_0^b \left(a^2 - \frac{a^2}{b}y\right)dy = \frac{a^2 b}{4}$$

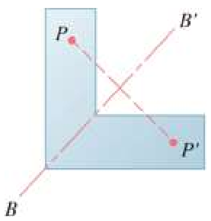
$$\boxed{\bar{x} = \frac{3}{4}a}$$

$$Q_x = \int \bar{y}_{el}\,dA = \int y(a-x)\,dy = \int y\left(a - \frac{a}{b^{1/2}}y^{1/2}\right)dy$$

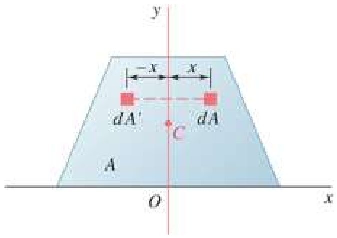
$$= \int_0^b \left(ay - \frac{a}{b^{1/2}}y^{3/2}\right)dy = \frac{ab^2}{10}$$

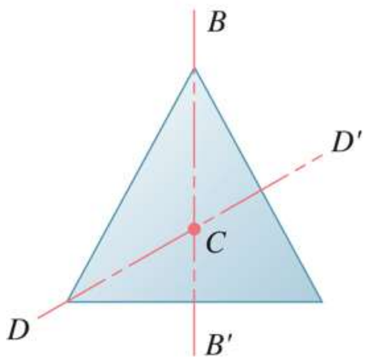
$$\boxed{\bar{y} = \frac{3}{10}b}$$

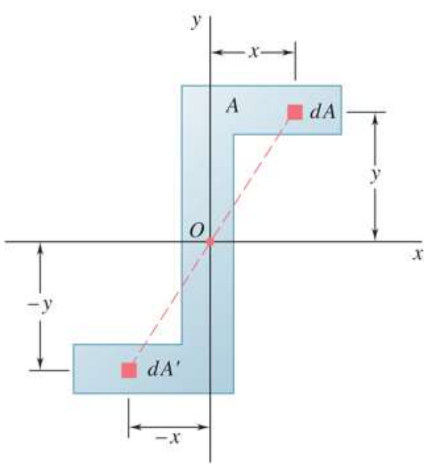
First Moments of Areas and Lines



(a)

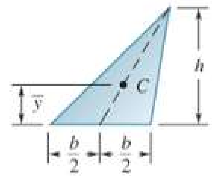
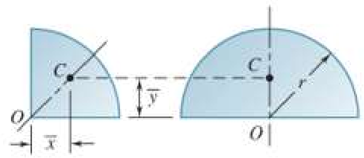
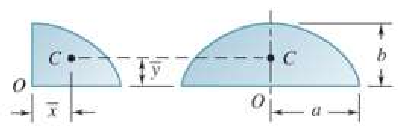
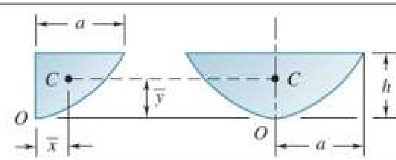
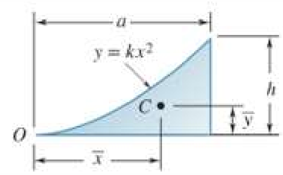
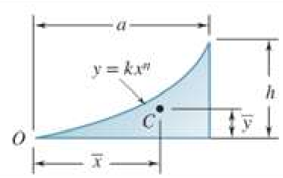
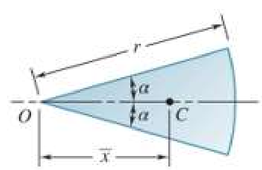




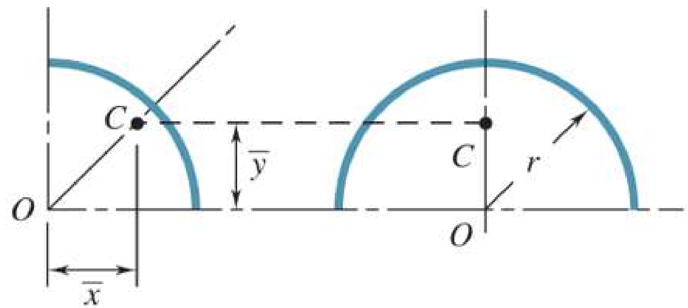
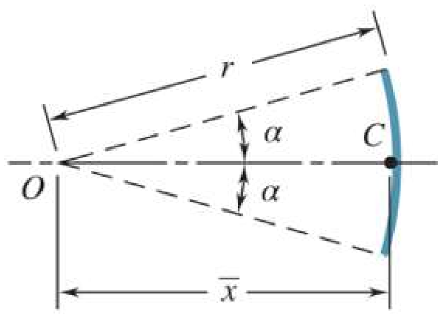


- An area is symmetric with respect to an axis *BB'* if for every point *P* there exists a point *P'* such that *PP'* is perpendicular to *BB'* and is divided into two equal parts by *BB'*.
- The first moment of an area with respect to a line of symmetry is zero.
- If an area possesses a line of symmetry, its centroid lies on that axis
- If an area possesses two lines of symmetry, its centroid lies at their intersection.
- An area is symmetric with respect to a center *O* if for every element *dA* at (*x*,*y*) there exists an area *dA'* of equal area at (-*x*,-*y*).
- The centroid of the area coincides with the center of symmetry.

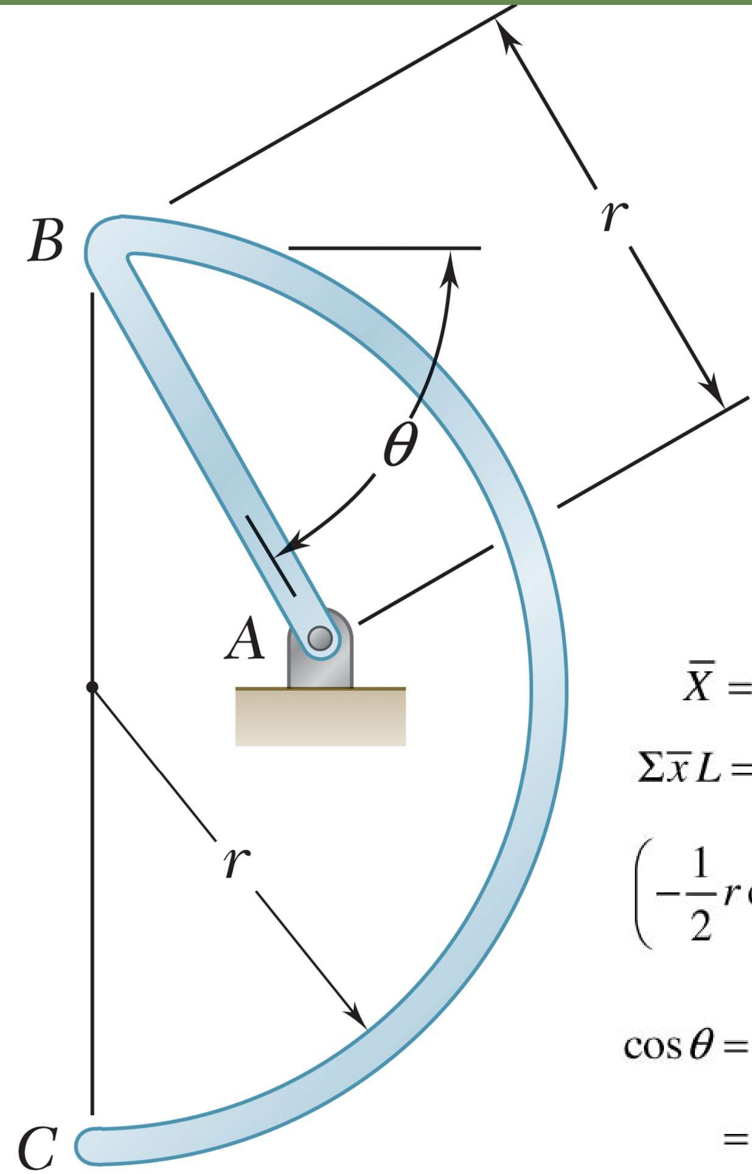
Centroids of Common Shapes of Areas

Shape		$\bar{x}$	$\bar{y}$	Area
Triangular area			$\frac{h}{3}$	$\frac{bh}{2}$
Quarter-circular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
Semicircular area		0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
Quarter-elliptical area		$\frac{4a}{3\pi}$	$\frac{4b}{3\pi}$	$\frac{\pi ab}{4}$
Semielliptical area		0	$\frac{4b}{3\pi}$	$\frac{\pi ab}{2}$
Semiparabolic area		$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
Parabolic area		0	$\frac{3h}{5}$	$\frac{4ah}{3}$
Parabolic spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$
General spandrel		$\frac{n+1}{n+2}a$	$\frac{n+1}{4n+2}h$	$\frac{ah}{n+1}$
Circular sector		$\frac{2r\sin\alpha}{3\alpha}$	0	αr^2

Centroids of Common Shapes of Lines

Shape		$\bar{x}$	$\bar{y}$	Length
Quarter-circular arc		$\frac{2r}{\pi}$	$\frac{2r}{\pi}$	$\frac{\pi r}{2}$
Semicircular arc		0	$\frac{2r}{\pi}$	πr
Arc of circle		$\frac{r \sin \alpha}{\alpha}$	0	$2\alpha r$

Problem 5.30



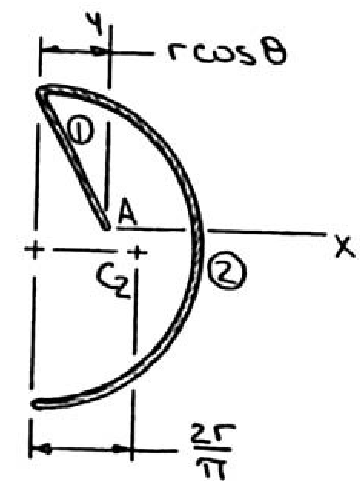
The homogeneous wire *ABC* is bent into a semicircular arc and a straight section as shown and is attached to a hinge at *A*. Determine the value of θ for which the wire is in equilibrium for the indicated position.

$$\bar{X} = 0$$

$$\Sigma \bar{x} L = 0$$

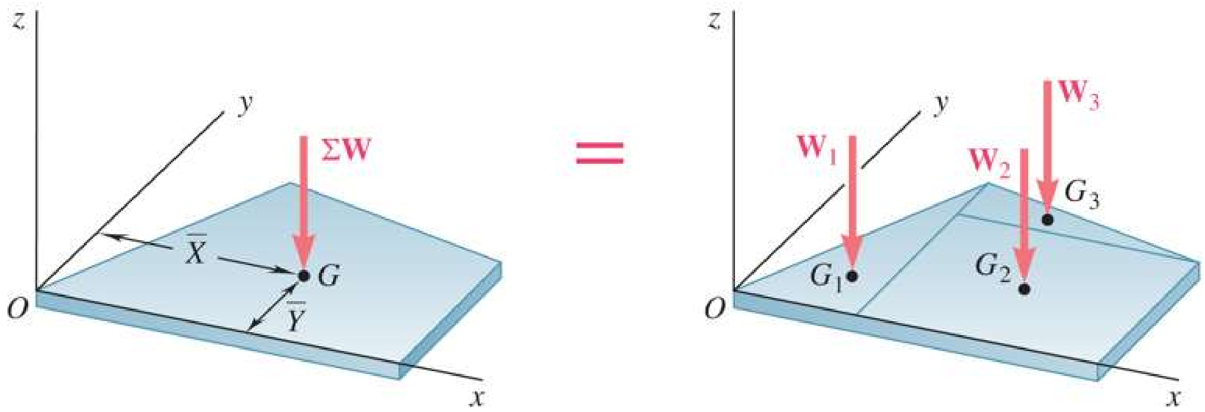
$$\left(-\frac{1}{2} r\cos\theta\right)(r) + \left(\frac{2r}{\pi} - r\cos\theta\right)(\pi r) = 0$$

$$\cos\theta = \frac{4}{1+2\pi}$$
$$= 0.54921$$



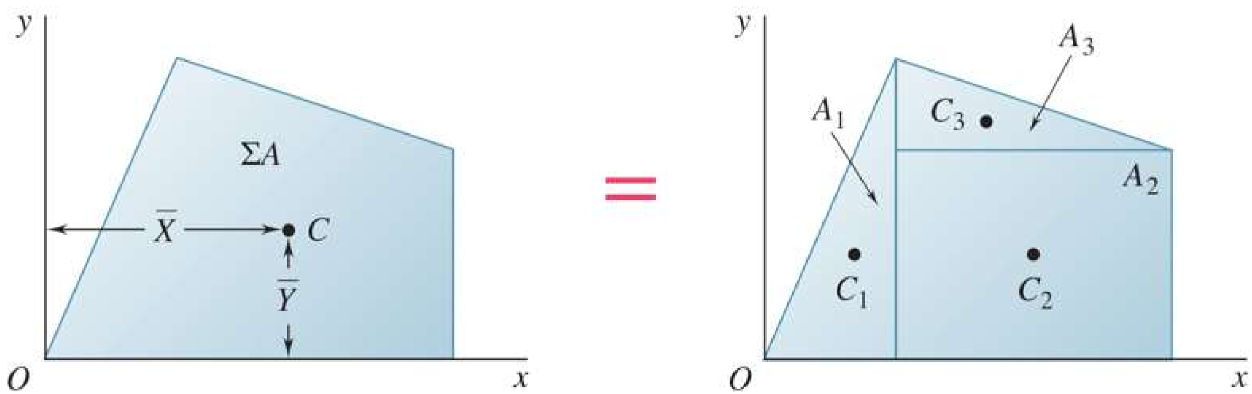
or $\theta = 56.7°$ ◀

Composite Plates and Areas



- Composite plates

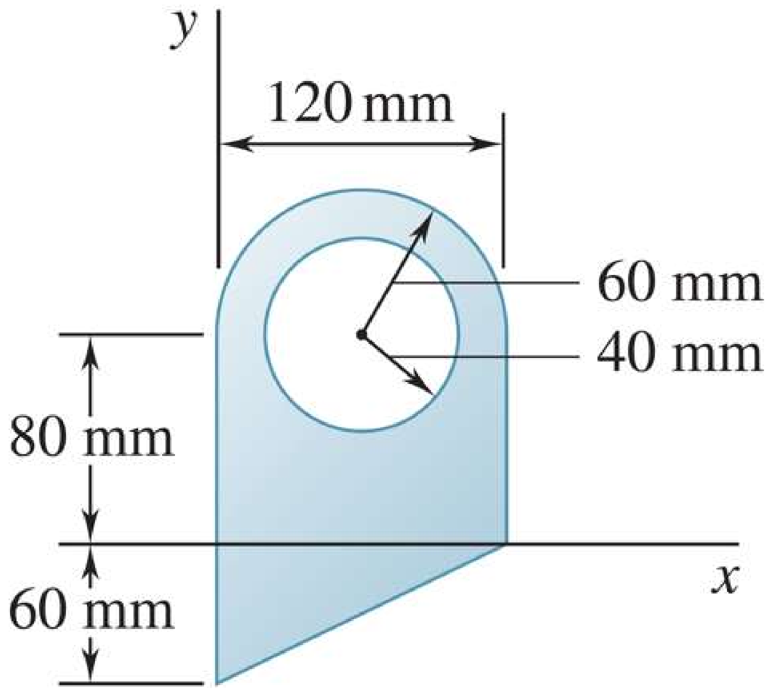
$$\bar{X}\sum W = \sum \bar{x} W$$
$$\bar{Y}\sum W = \sum \bar{y} W$$



- Composite area

$$\bar{X}\sum A = \sum \bar{x} A$$
$$\bar{Y}\sum A = \sum \bar{y} A$$

Sample Problem 5.1

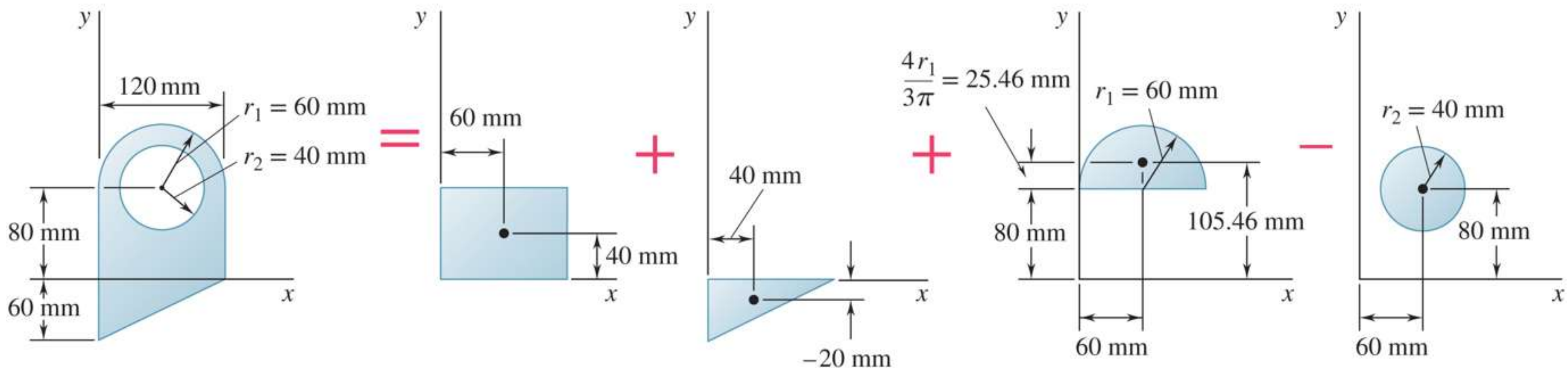


For the plane area shown, determine the first moments with respect to the *x* and *y* axes and the location of the centroid.

SOLUTION:

- Divide the area into a triangle, rectangle, and semicircle with a circular cutout.
- Calculate the first moments of each area with respect to the axes.
- Find the total area and first moments of the triangle, rectangle, and semicircle. Subtract the area and first moment of the circular cutout.
- Compute the coordinates of the area centroid by dividing the first moments by the total area.

Sample Problem 5.1



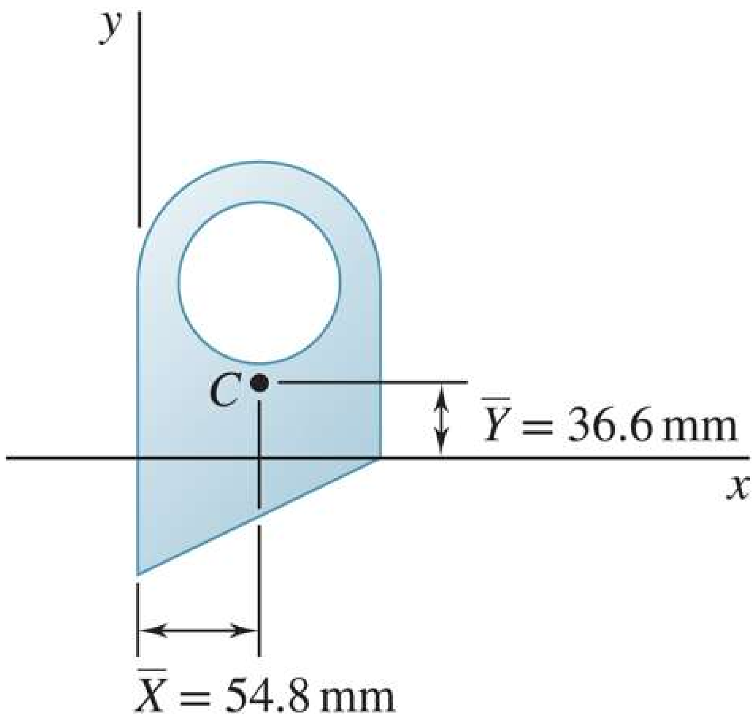
Component	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
Rectangle	$(120)(80) = 9.6 \times 10^3$	60	40	$+576 \times 10^3$	$+384 \times 10^3$
Triangle	$\frac{1}{2}(120)(60) = 3.6 \times 10^3$	40	−20	$+144 \times 10^3$	-72×10^3
Semicircle	$\frac{1}{2}\pi(60)^2 = 5.655 \times 10^3$	60	105.46	$+339.3 \times 10^3$	$+596.4 \times 10^3$
Circle	$-\pi(40)^2 = -5.027 \times 10^3$	60	80	-301.6×10^3	-402.2×10^3
	$\Sigma A = 13.828 \times 10^3$			$\Sigma\bar{x}A = +757.7 \times 10^3$	$\Sigma\bar{y}A = +506.2 \times 10^3$

- Find the total area and first moments of the triangle, rectangle, and semicircle. Subtract the area and first moment of the circular cutout.

$$\boxed{\begin{aligned} Q_x &= +506.2\times10^3\,\text{mm}^3 \\ Q_y &= +757.7\times10^3\,\text{mm}^3 \end{aligned}}$$

Sample Problem 5.1

- Compute the coordinates of the area centroid by dividing the first moments by the total area.



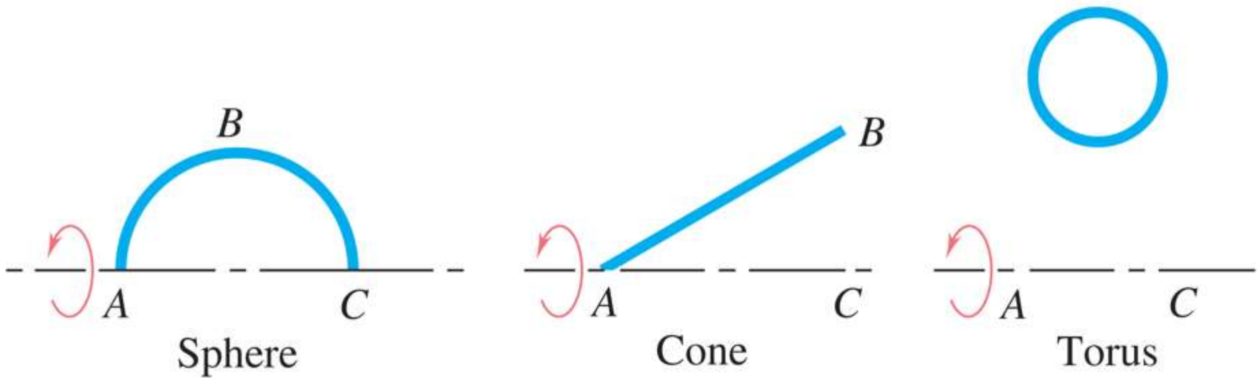
$$\bar{X} = \frac{\sum \bar{x} A}{\sum A} = \frac{+757.7 \times 10^3 \text{ mm}^3}{13.828 \times 10^3 \text{ mm}^2}$$

$$\boxed{\bar{X} = 54.8 \text{ mm}}$$

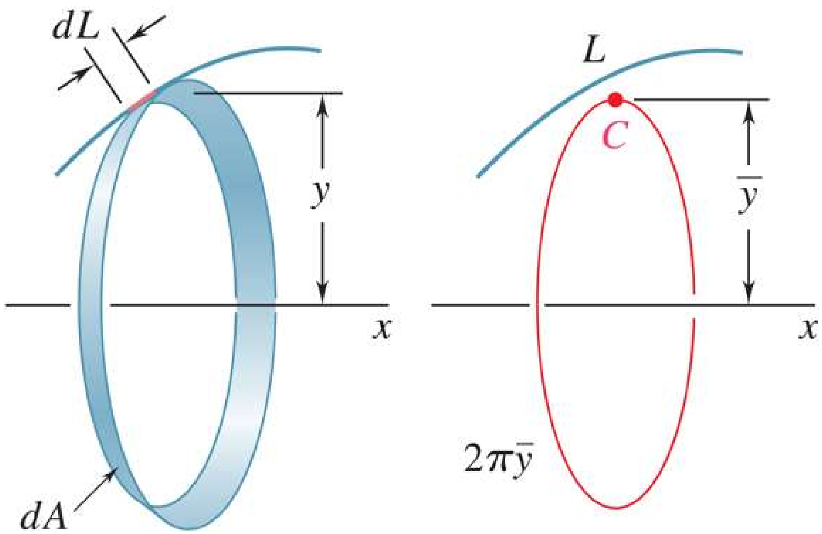
$$\bar{Y} = \frac{\sum \bar{y} A}{\sum A} = \frac{+506.2 \times 10^3 \text{ mm}^3}{13.828 \times 10^3 \text{ mm}^2}$$

$$\boxed{\bar{Y} = 36.6 \text{ mm}}$$

Theorems of Pappus-Guldinus



- Surface of revolution is generated by rotating a plane curve about a fixed axis.

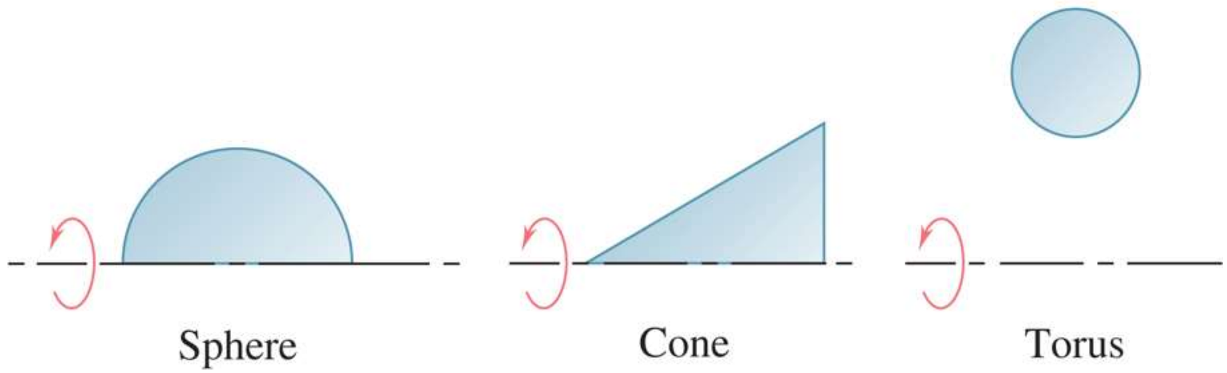


THEOREM I:

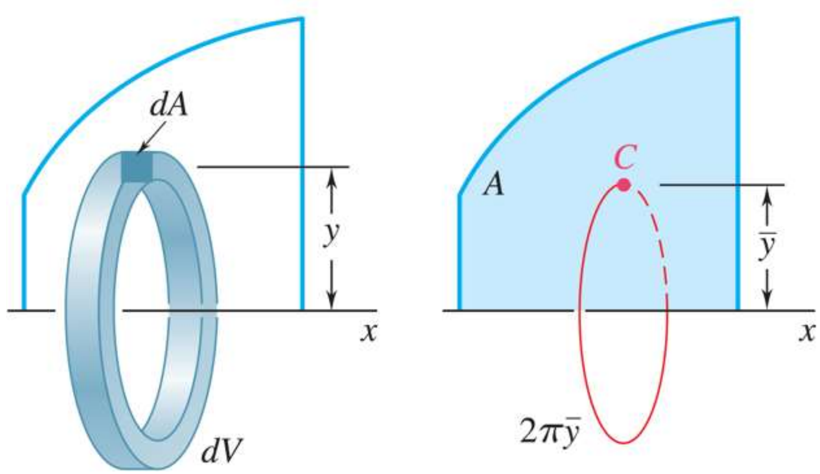
Area of a surface of revolution is equal to the length of the generating curve times the distance traveled by the centroid through the rotation.

$$A = 2\pi \bar{y} L$$

Theorems of Pappus-Guldinus



- Body of revolution is generated by rotating a plane area about a fixed axis.

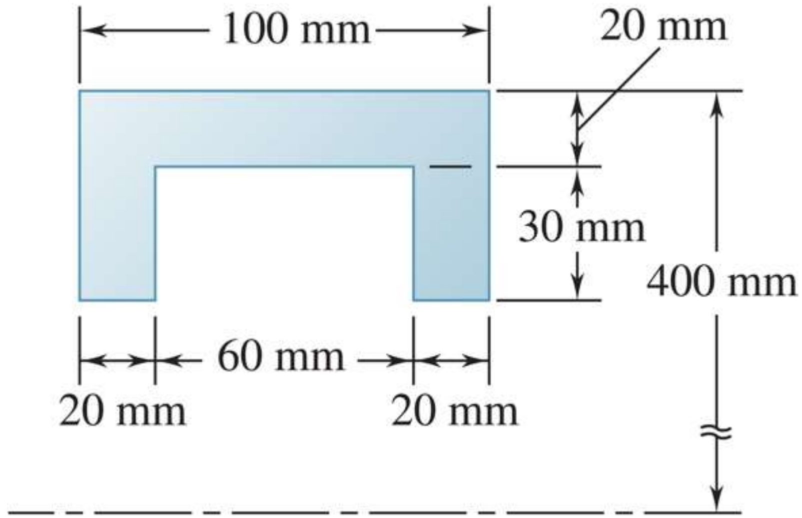


THEOREM II:

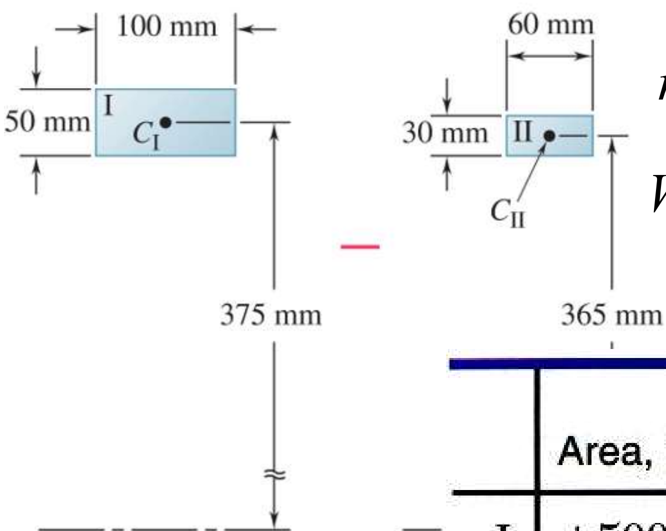
Volume of a body of revolution is equal to the generating area times the distance traveled by the centroid through the rotation.

$$V = 2\pi \bar{y} A$$

Sample Problem 5.7



The outside diameter of a pulley is 0.8 m, and the cross section of its rim is as shown. Knowing that the pulley is made of steel and that the density of steel is $\rho = 7.85\times10^3\ \mathrm{kg/m^3}$ determine the mass and weight of the rim.

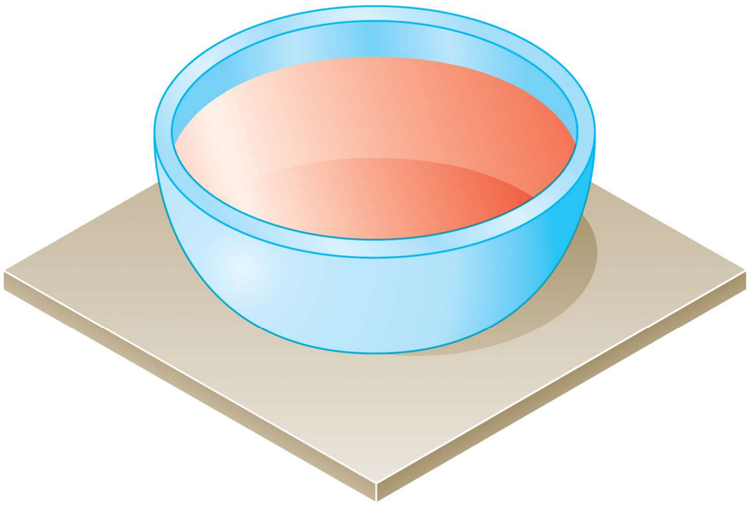


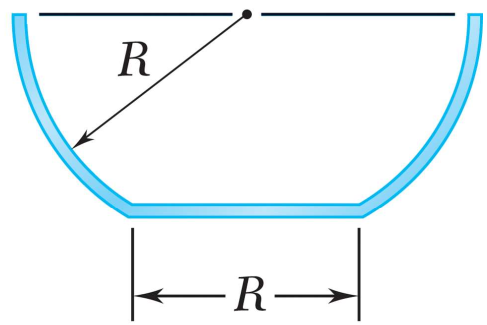
$$m = \rho V = \left(7.85\times10^3\ \mathrm{kg/m^3}\right)\left(7.65\times10^6\ \mathrm{mm^3}\right)\left(10^{-9}\ \mathrm{m^3/mm^3}\right)$$

$$W = mg = \left(60.0\ \mathrm{kg}\right)\left(9.81\ \mathrm{m/s^2}\right) \quad \boxed{m = 60.0\ \mathrm{kg}} \quad \boxed{W = 589\ \mathrm{N}}$$

	Area, mm^2	$\bar{y}$, mm	Distance Traveled by C, mm	Volume, mm^3
I	+5000	375	$2\pi(375) = 2356$	$(5000)(2356) = 11.78 \times 10^6$
II	−1800	365	$2\pi(365) = 2293$	$(-1800)(2293) = -4.13 \times 10^6$
				Volume of rim $= 7.65 \times 10^6$

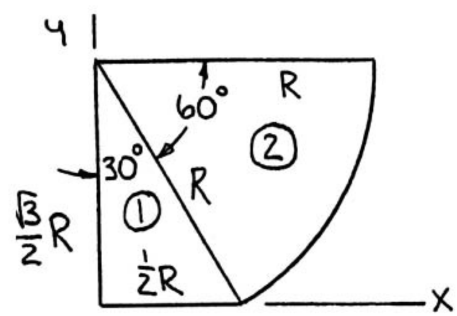
Problem 5.64





Determine the capacity, in liters, of the punch bowl shown if R = 250 mm.

$$V = 2\pi\bar{x}A = 2\pi\Sigma\bar{x}A$$
$$= 2\pi(\bar{x}_1A_1 + \bar{x}_2A_2)$$
$$= 2\pi\left[\left(\frac{1}{3}\times\frac{1}{2}R\right)\left(\frac{1}{2}\times\frac{1}{2}R\times\frac{\sqrt{3}}{2}R\right) + \left(\frac{2R\sin 30^\circ}{3\times\frac{\pi}{6}}\right)\left(\frac{\pi}{6}R^2\right)\right]$$
$$= 2\pi\left(\frac{R^3}{16\sqrt{3}} + \frac{R^3}{2\sqrt{3}}\right)$$
$$= \frac{3\sqrt{3}}{8}\pi R^3$$
$$= \frac{3\sqrt{3}}{8}\pi(0.25\ \text{m})^3$$
$$= 0.031883\ \text{m}^3$$

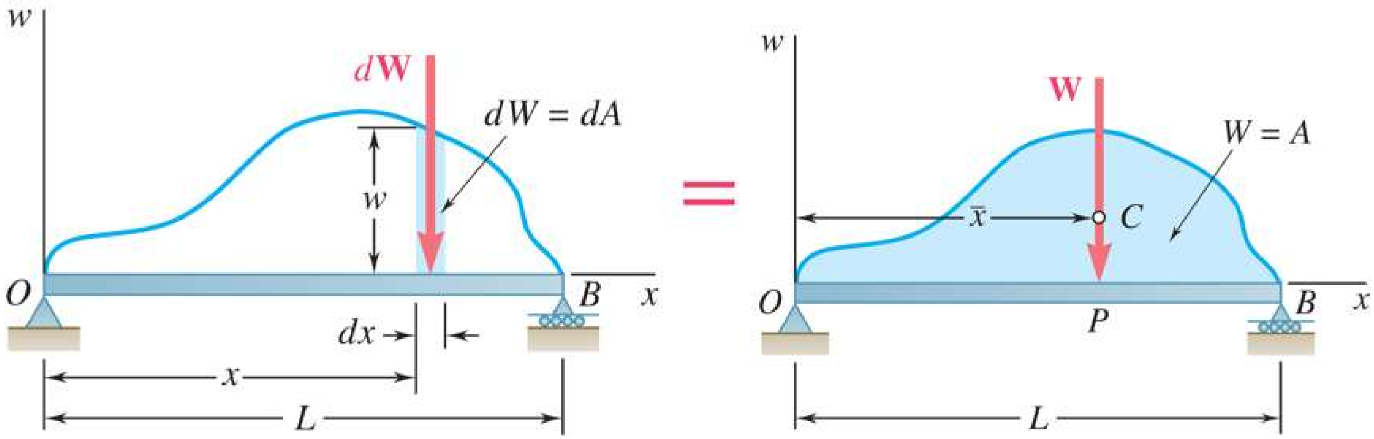


$$10^3\ \text{l} = 1\ \text{m}^3$$

$$V = 0.031883\ \text{m}^3 \times \frac{10^3\ \text{l}}{1\ \text{m}^3}$$

$V = 31.9\ \text{l}$ ◀

Distributed Loads on Beams



$$W = \int_0^L w\,dx = \int dA = A$$

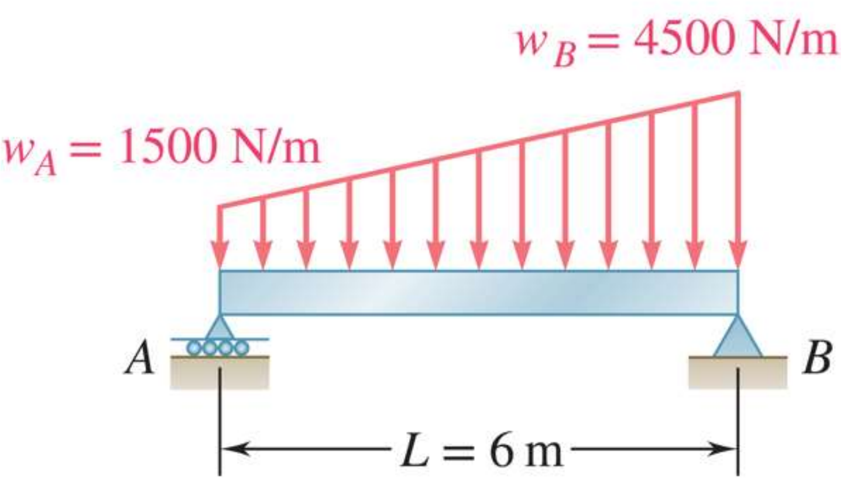
- A distributed load is represented by plotting the load per unit length, *w* (N/m) . The total load is equal to the area under the load curve.

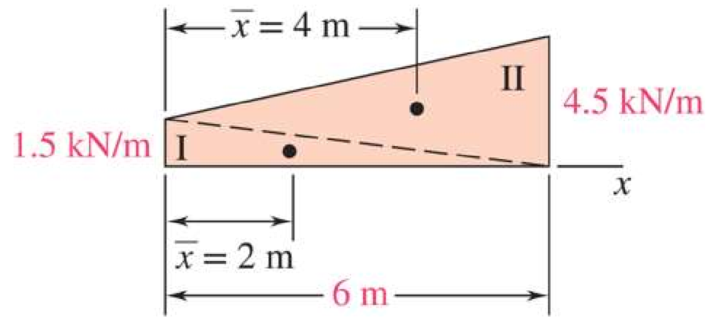
$$(OP)W = \int x\,dW$$

$$(OP)A = \int_0^L x\,dA = \bar{x}A$$

- A distributed load can be replace by a concentrated load with a magnitude equal to the area under the load curve and a line of action passing through the area centroid.

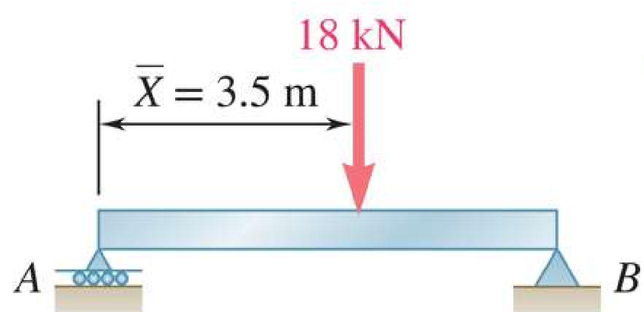
Sample Problem 5.9



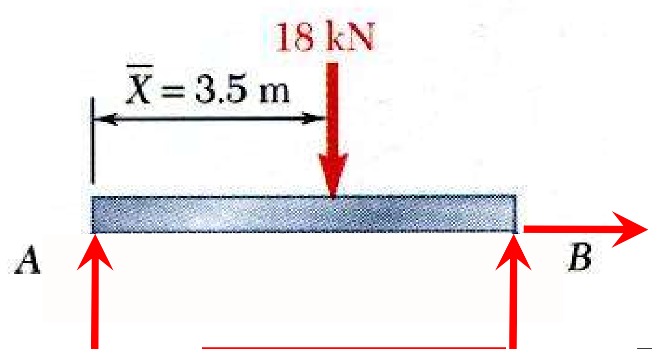


$\boxed{F = 18.0\ \text{kN}}$

$\boxed{\bar{X} = 3.5\ \text{m}}$

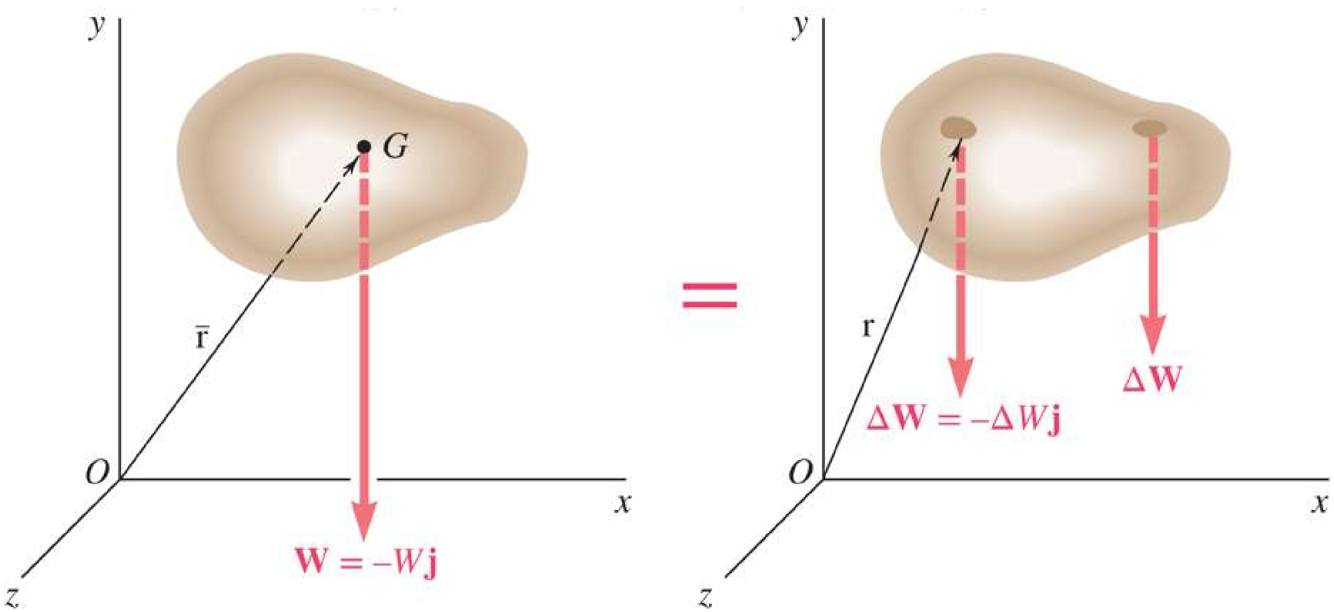


A beam supports a distributed load as shown. Determine the equivalent concentrated load and the reactions at the supports.



$\boxed{B_y = 10.5\ \text{kN}}$ $\boxed{A_y = 7.5\ \text{kN}}$ $B_x = 0$

Center of Gravity of a 3D Body: Centroid of a Volume



- Center of gravity G

$$-W\vec{j} = \sum\left(-\Delta W\vec{j}\right)$$

$$\vec{r}_G \times \left(-W\vec{j}\right) = \sum\left[\vec{r} \times \left(-\Delta W\vec{j}\right)\right]$$
$$\vec{r}_G W \times \left(-\vec{j}\right) = \left(\sum \vec{r}\Delta W\right) \times \left(-\vec{j}\right)$$

$$W = \int dW \qquad \vec{r}_G W = \int \vec{r}\, dW$$

- Results are independent of body orientation,

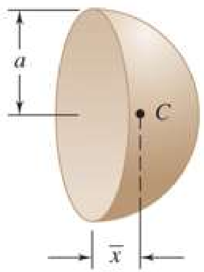
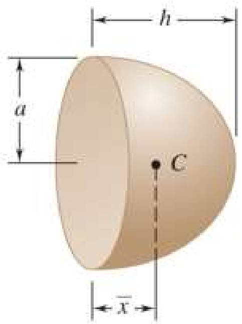
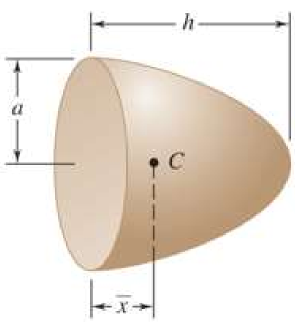
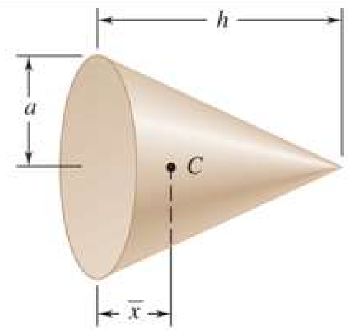
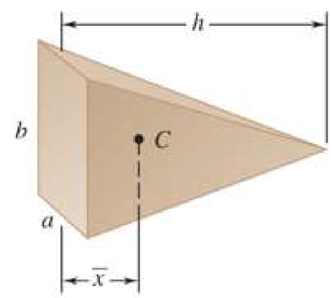
$$\bar{x}W = \int x\,dW \quad \bar{y}W = \int y\,dW \quad \bar{z}W = \int z\,dW$$

- For homogeneous bodies,

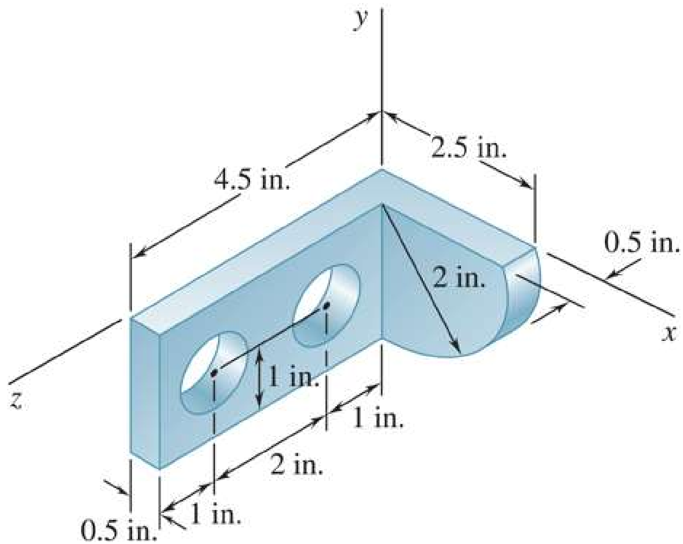
$$W = \gamma V \text{ and } dW = \gamma\, dV$$

$$\bar{x}V = \int x\,dV \quad \bar{y}V = \int y\,dV \quad \bar{z}V = \int z\,dV$$

Centroids of Common 3D Shapes

Shape		$\bar{x}$	Volume
Hemisphere		$\frac{3a}{8}$	$\frac{2}{3}\pi a^3$
Semiellipsoid of revolution		$\frac{3h}{8}$	$\frac{2}{3}\pi a^2 h$
Paraboloid of revolution		$\frac{h}{3}$	$\frac{1}{2}\pi a^2 h$
Cone		$\frac{h}{4}$	$\frac{1}{3}\pi a^2 h$
Pyramid		$\frac{h}{4}$	$\frac{1}{3}abh$

Composite 3D Bodies

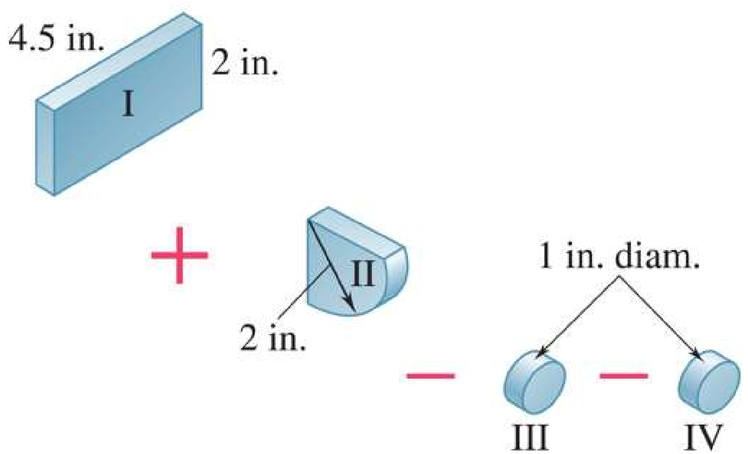


- Moment of the total weight concentrated at the center of gravity G is equal to the sum of the moments of the weights of the component parts.

$$\bar{X}\sum W = \sum \bar{x}W \quad \bar{Y}\sum W = \sum \bar{y}W \quad \bar{Z}\sum W = \sum \bar{z}W$$

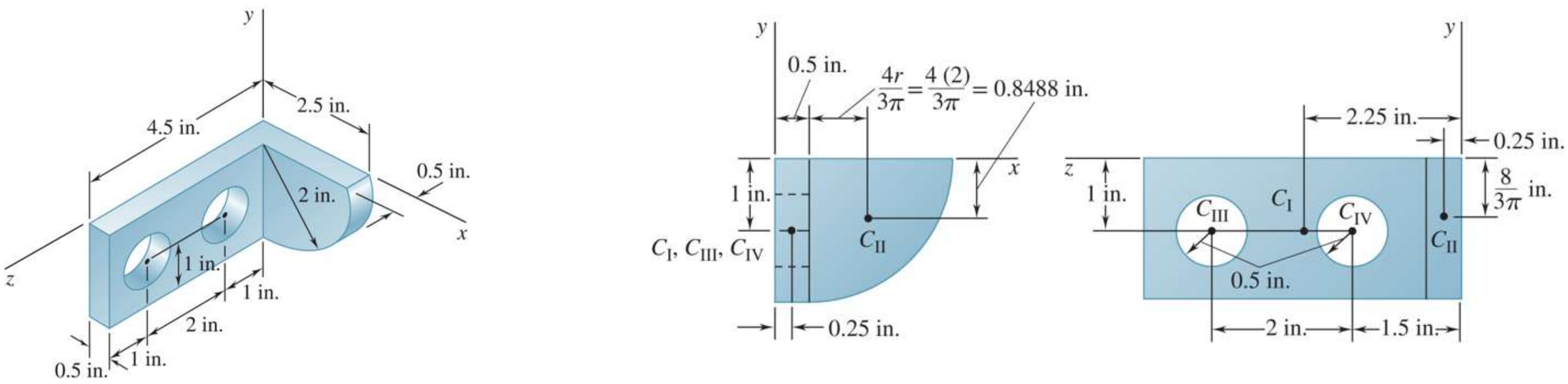
- For homogeneous bodies,

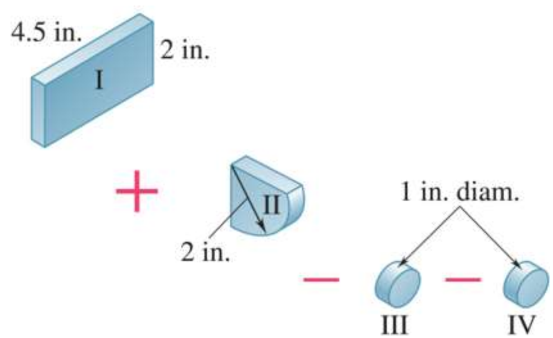
$$\bar{X}\sum V = \sum \bar{x}V \quad \bar{Y}\sum V = \sum \bar{y}V \quad \bar{Z}\sum V = \sum \bar{z}V$$



Sample Problem 5.12

Modeling:





	V, in³	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{z}$, in.	$\bar{x}V$, in⁴	$\bar{y}V$, in⁴	$\bar{z}V$, in⁴
I	$(4.5)(2)(0.5) = 4.5$	0.25	−1	2.25	1.125	−4.5	10.125
II	$\frac{1}{4}\pi(2)^2(0.5) = 1.571$	1.3488	−0.8488	0.25	2.119	−1.333	0.393
III	$-\pi(0.5)^2(0.5) = -0.3927$	0.25	−1	3.5	−0.098	0.393	−1.374
IV	$-\pi(0.5)^2(0.5) = -0.3927$	0.25	−1	1.5	−0.098	0.393	−0.589
	$\Sigma V = 5.286$				$\Sigma\bar{x}V = 3.048$	$\Sigma\bar{y}V = -5.047$	$\Sigma\bar{z}V = 8.555$

Sample Problem 5.12

	V, in³	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{z}$, in.	$\bar{x}V$, in⁴	$\bar{y}V$, in⁴	$\bar{z}V$, in⁴
I	$(4.5)(2)(0.5) = 4.5$	0.25	-1	2.25	1.125	-4.5	10.125
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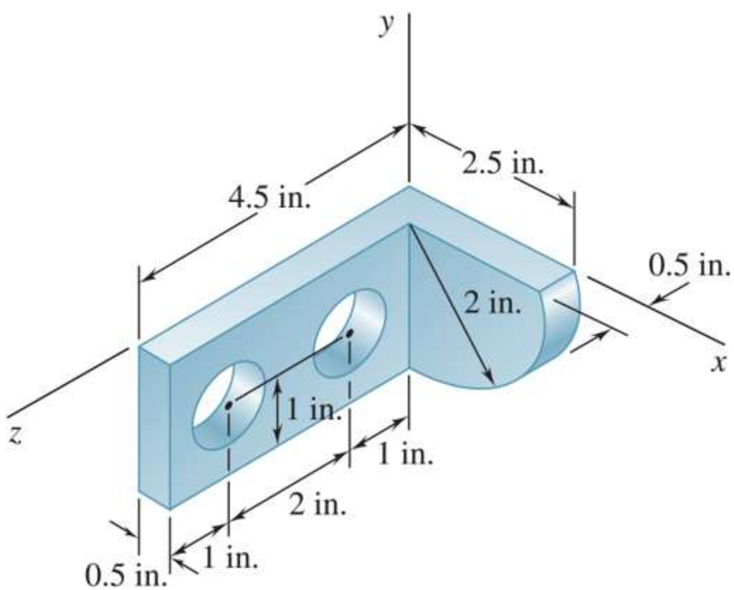
	$\Sigma V = 5.286$				$\Sigma \bar{x}V = 3.048$	$\Sigma \bar{y}V = -5.047$	$\Sigma \bar{z}V = 8.555$

Analysis:



$$\bar{X} = \sum \bar{x}V \Big/ \sum V = \left(3.048 \text{ in}^4\right) \Big/ \left(5.286 \text{ in}^3\right)$$

$$\boxed{\bar{X} = 0.577 \text{ in.}}$$

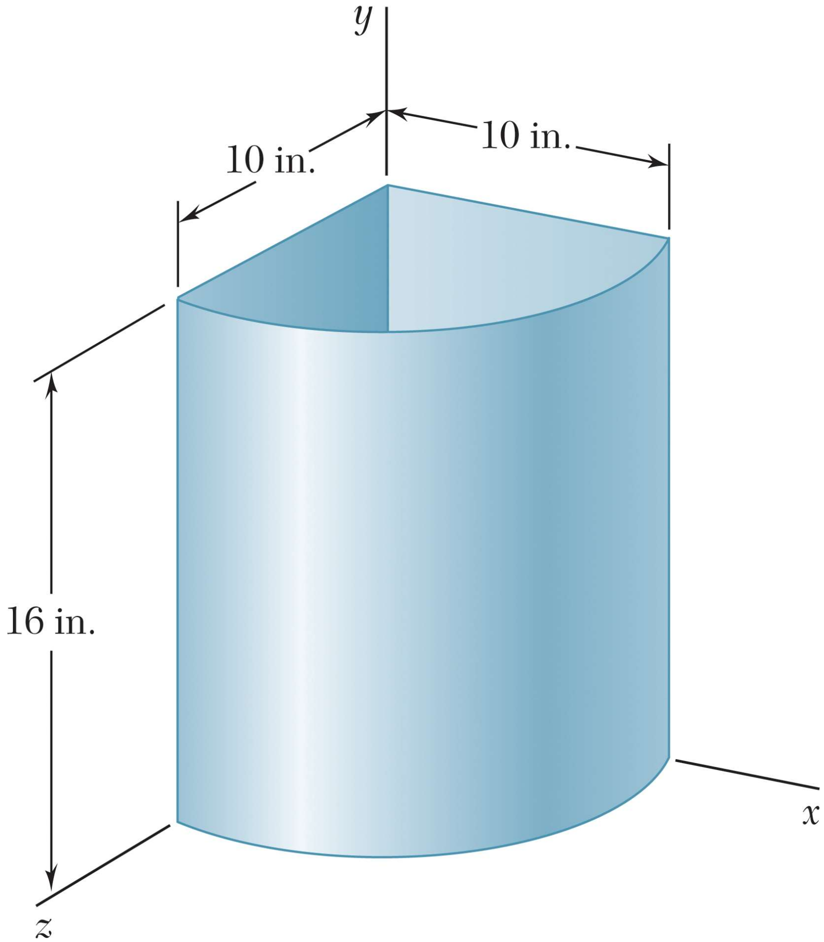
$$\bar{Y} = \sum \bar{y}V \Big/ \sum V = \left(-5.047 \text{ in}^4\right) \Big/ \left(5.286 \text{ in}^3\right)$$

$$\boxed{\bar{Y} = -0.955 \text{ in.}}$$

$$\bar{Z} = \sum \bar{z}V \Big/ \sum V = \left(8.555 \text{ in}^4\right) \Big/ \left(5.286 \text{ in}^3\right)$$

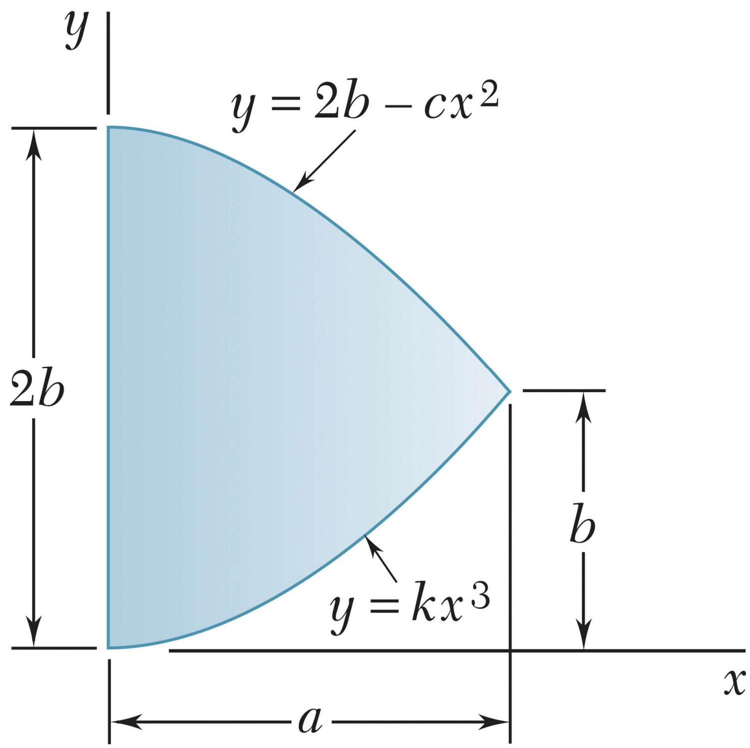
$$\boxed{\bar{Z} = 1.618 \text{ in.}}$$

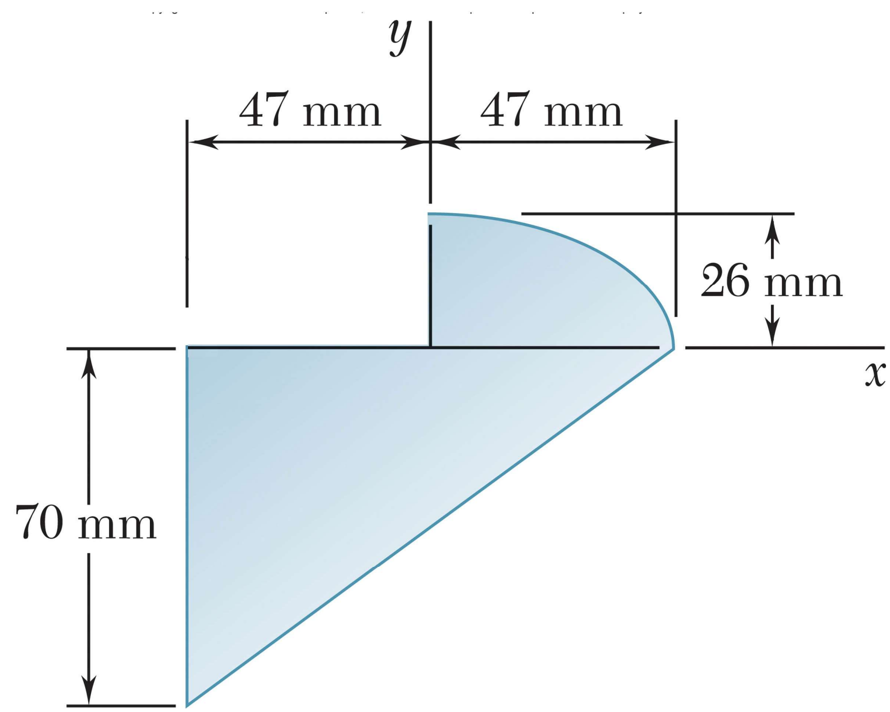
Problem 5.110



A wastebasket, designed to fit in the corner of a room, is 16 in. high and has a base in the shape of a quarter circle of radius 10 in. Locate the center of gravity of the wastebasket, knowing that it is made of sheet metal of uniform thickness.

Problem 5. 41 5.15





Determine by direct integration the centroid of the area shown. Express your answer in terms of *a* and *b*.

Locate the centroid of the plane area, which consists of a quarter-ellipse and a triangle.